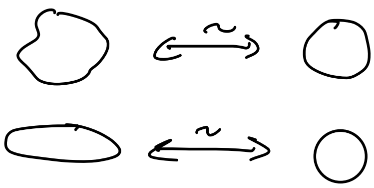
Equivariant topology concerns itself with topological spaces posessing symmetries, and continuous maps that respect them.

Objects: Top. Spaces w/ cont. action by a group G.

Morphisms: cont. maps $f: X \longrightarrow Y$ such that

$$f(g \cdot x) = g \cdot f(x)$$

i.e.:



topologically

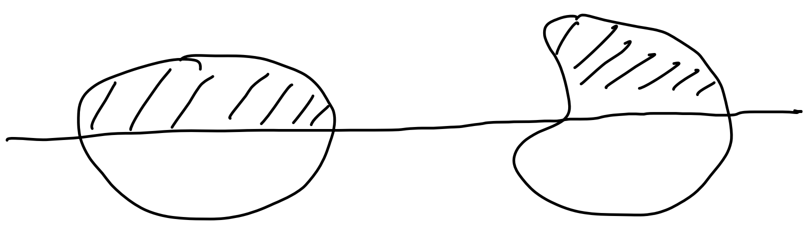
equivariantly with $\mathbb{Z}/2$ action given by reflection.

<u>Key Goals:</u>

1) Demonstrate how to reduce some interesting (?) geometric problems to students of equivariant topology

2) Introduce some topological tools we can use.

<u>(2D) Ham Sandwich Theorem</u>: Given 2 $\boxed{\text{masses}}$ μ_1, μ_2 on $\mathbb{R}^2$ plane, there exists a line equiparting them.



A <u>mass</u> is a positive finite borel measure where hyperplanes have measure zero

$A \subset \mathbb{R}^2$ compact

$$\boxed{\mu(\mathbb{R}^2) = \int_{\mathbb{R}^n} \chi_A \, d\mu}$$

* Rmk: Like equipartitioning 1 cross is intermediate value theorem.

Strategy:

Employ the Borsuk-Ulam theorem:

i.e. $\forall$ cont. map $f: S^2 \longrightarrow \mathbb{R}^2$

$\exists$ some $x \in S^2 \ni f(x) = f(-x)$.

Equivariant version: Every cont. map

$f: S^2 \longrightarrow \mathbb{R}^2$ where $f(-x) = -f(x)$

has a zero [Equivalent via $g(x) := f(x) - f(-x)$].

Here $\mathbb{Z}/2 \curvearrowright S^2$ antipodally & $\mathbb{R}^2$ $(-1)(x,y) = (-x,-y)$.

How: Topologize space of oriented lines $ax+by=c$ is a line if a,b not simultaneously 0.

Scale so that $(a,b,c) \in S^2$

consider "half-space" determined by line:

$\forall$ $(a,b,c) \in S^2$, define

$$h^+(\vec{u}) := \{(x,y) \in \mathbb{R}^2 \mid ax+by \geq c\}$$

Note that we take $h^+((0,0,1)) = \mathbb{R}^2$, $h^+((0,0,-1)) = \emptyset$

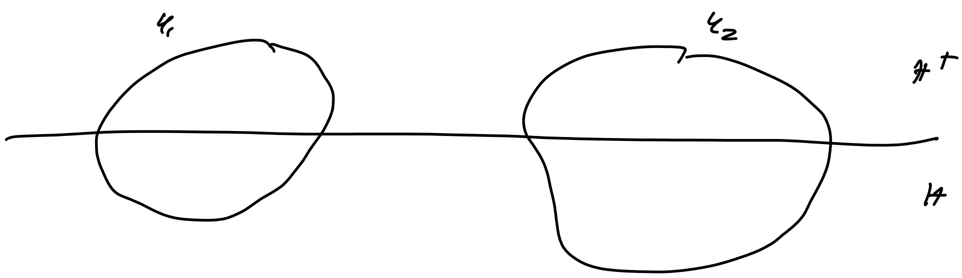
"lines at infinity"

Similarly,

$$H^{-1}(\vec{u}) := \{(x,y) \in \mathbb{R}^2 : ax + by \le c\}$$

Define $f: S^2 \longrightarrow \mathbb{R}^2$ as an "Area" function,

$$f(\vec{x}) = \left(\mu_1(H^+(\vec{x})) - \mu_1(H^-(\vec{x})),\ \mu_2(H^+(\vec{x})) - \mu_2(H^-(\vec{x}))\right)$$



Notice that

$$H^+(-a,-b,-c) = \{(x,y) \in \mathbb{R}^2 \mid ax + by \ge c\} = H^-(a,b,c)$$

$\Rightarrow f(-\vec{x}) = -f(\vec{x})$. (Borsuk–Ulam.)

Let's generalize this.

<u>The configuration space-test map set up.</u>

Step 1: Form a configuration space X of all possible geometric arrangements

($X = S^2$, oriented lines)

Step 2: Find a natural test space Y

($Y = \mathbb{R}^2$, ordered pairs of Areas)

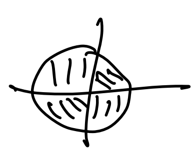
Step 3: If both X, Y have "obvious" symmetries given by an action G, take an equivariant map $f: X \to Y$

Step 4: A solution space $Z \subset Y$
$((0,0) \in \mathbb{R}^2)$

Step 5: Show that an equivariant map
$f: X \xrightarrow[G]{} Y/Z$ is impossible
(Borsuk-Ulam.)

Let's consider how we can consider mild generalizations of ham-sandwich type theorems. For example:

Q_1: Given 3 masses μ_1, μ_2, μ_3 and 2 hyperplanes, what is the minimal dimension d such that we can guarantee an equipartition?



$$f: (S^d \times S^d) \longrightarrow ((\mathbb{R}^2)^2)^3 = \mathbb{R}^{12}$$

$$f(H_1, H_2) = \left(\mu_j(H_1 \cap H_2) - \frac{1}{4}\mu_j(\mathbb{R}^d)\right)_{\substack{j = \{1,2,3\} \\ \beta \in \{+,-\}^2}}$$

Q_2: When can we guarantee that the planes are orthogonal? Add a function $g: (S^d \times S^d) \longrightarrow \mathbb{R}$
$(a, b) \longmapsto \langle a, b \rangle$
etc.

We need a more robust approach to solve harder problems.

The idea is to replace an equivariant map $f : X \longrightarrow_G Y$ with a section

$$s_f : X/G \longrightarrow (X\times Y)/G : [x] \longmapsto [x, f(x)]$$

where $X \times Y$ gets diagonal action $g(x,y) = (gx, gy)$, and the bundle is $p : (X\times Y)/G \longrightarrow X/G$

G compact, X, Y Hausdorff, correspondence

$\mu(X,Y) = \{[x,y] \in (X\times Y)/G \mid G_x \subset G_y\}$

Remark: If G does not act freely on X, then this is not well-defined, but we can instead define a section on

$$EG \times_G X \longrightarrow EG \times_G (X \times Y)$$

i.e.: Replace construction with "homotopically equivalent situation."

top arrow equivalence by LES homotopy

$$\begin{tikzcd}
EG \times_G (X\times Y) \arrow[r, "pr"] \arrow[d, "p'"'] & X \times_G Y \arrow[d, "p"] \\
EG \times_G X \arrow[r] & X/G
\end{tikzcd}$$

Why is this easier?

Well in our case $Y := \mathbb{R}^N$ and $(\mathbb{Z}/2)^k$ acts linearly. Hence

$$p : (X \times \mathbb{R}^n)/G \longrightarrow X/G$$

is actually a vector bundle and we can appeal to the theory of characteristic classes to show nonexistence of characteristic classes.

Remark: this happens a lot since in geometric configurations you end up with $\mathbb{R}^n$ as a test space since we are concerned with lengths, areas, angles, etc.

Key Theorem: $p : E \to B$ a v.b. of rank n admitting a nonvanishing section. Then o n^{th} Stiefel-Whitney class $w_n(p) \in H^n(B, \mathbb{Z}/2)$ is trivial.

On computability: the key computational facts used to prove these theorems is the fact $(\mathbb{Z}/2)^k$ is abelian, reps are 1 dimensional, & $\mathbb{R}^n$ decomposes accordingly. Whitney sum formula lets you calculate w_n

Results: Given m masses, h hyperplanes, let $\Delta(d,h)$ be the minimal dimension such that we can guarantee an equipartition.

Thm: (Levitska, Živaljević, Vrećica): $\Delta(h, m = 2^{q} + r) \leq 2^{h+q-1} + r$

very few explicitly known sharp results.

We can also generalize the Borsuk-Ulam theorem in a slightly different way:

To solve this we will need "index theory", which will be a measure of "G-complexity"

OPTIONAL DISCUSSION

Def: Let G be a finite group, $|G| > 1$ and $n \in \mathbb{N}_0$.

A G-space X is an $E_n G$ space if it satisfies

1: X is a free G-space

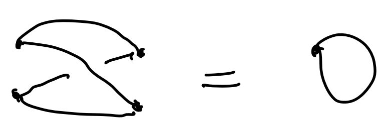
2: X is a finite CW (or simplicial) complex

3: X is $(n-1)$-connected ($\pi_i(X) = 1$ for $i \leq n-1$).

Milnor showed that these exist using topological join iteratively

$$X * Y := (X \times I \times Y)/\sim$$

$$(x, 0, y_1) \sim (x, 0, y_2),$$
$$(x_1, 1, y) \sim (x_2, 1, y).$$

ex: $S^1 = \mathbb{Z}_2 * \mathbb{Z}_2$ 

$S^2 =$

$\mathbb{Z}/2 \curvearrowright S^n$ is $E_n\,\mathbb{Z}/2$

Classical:

Thm: K a finite simplicial complex

s.t. a finite group G acts freely on.

If X is an $(n-1)$-connected, then

$\exists\ f: K \longrightarrow_G X$.

In particular, $\exists\ f: X \longrightarrow_G Y$ for any

E_nG spaces X, Y.

Generalized Borsuk-Ulam theorem

Thm 2: No G-equivariant map

from an E_nG space to

$E_{n-1}G$ space.

$G = \mathbb{Z}/2$ means this is Borsuk-Ulam

as well!

Now, we can define:

$$\mathrm{ind}_G(X) := \{\min(\mathbb{N}_0) : \exists\ f: X \longrightarrow_G E_nG\}$$

Properties:

1) (monotonicity) if $\exists\ f: X \longrightarrow_G Y \Rightarrow \mathrm{ind}_G(X) \leq \mathrm{ind}_G(Y)$

2) $\mathrm{ind}_G(E_nG) = n$

3) X is $(n-1)$-connected $\Rightarrow \mathrm{ind}_G(X) \geq n$

<u>Dold's Theorem</u>: Let G be a finite nontrivial group. If X is an n-connected G-space (CW/simplicial), and Y is a free simplicial/cellular complex with $\dim Y \leq n$, then there is no G-equivariant map from $X \to Y$

Pf: $\mathrm{ind}_G(X) \geq n+1$ while $\mathrm{ind}_G(Y) \leq n$. By monotonicity, this is impossible.

Theorem: let p be an odd prime, $n \in \mathbb{N}$, $p < n$, let $A_1, \dots, A_p$ the vertices of a regular polygon with p-sides on a great circle of S^{n-1}. Then $\forall$ continuous $f : S^{n-1} \longrightarrow \mathbb{R}$, $\exists$ $\rho \in SO(n)$. such that $f(\rho(A_1)) = f(\rho(A_2)) = \dots = f(\rho(A_p))$.

pf sketch: let $X_1, \dots, X_p$ be such points. Then the tuple is determined by X_1, X_2 alone, & likewise any vectors w/ angle $2\pi/p$ determine another $X_3, \dots, X_p$.

So, our configuration space can be identified with $V_2(\mathbb{R}^n)$.

Our test map is

$$F(X_1, \dots, X_n) := (f(X_1), \dots, f(X_p)).$$

Our test Space is then $\mathbb{R}^p$ with Solution Space $\Delta := \{(x_1, \ldots, x_p) : x_1 = \cdots = x_p\}$

$\mathbb{Z}/p$ acts on both spaces by cyclic permutation & f is equivariant. $\mathbb{Z}/p \curvearrowright V_2(\mathbb{R}^p)$ is free

i.e: $\underline{1}\,(x_1, \ldots, x_p) = (x_2, x_3, \ldots, x_p, x_1)$

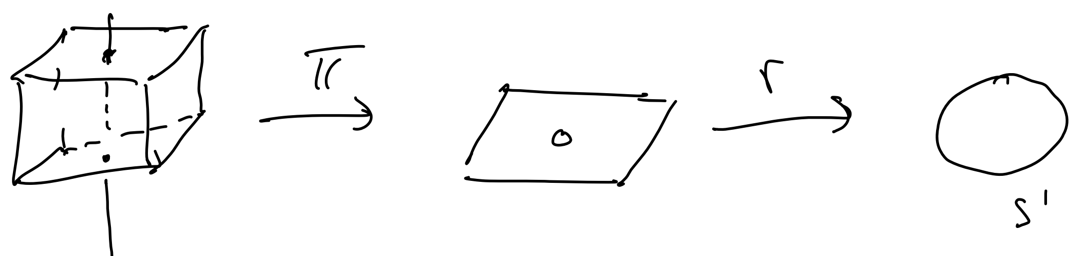
[Recall identification $V_2\,\mathbb{R}^n$]

Suppose there were ~~an~~ a $\mathbb{Z}/p$-equivariant map

$$f : V_2(\mathbb{R}^n) \xrightarrow[\mathbb{Z}/p]{} \mathbb{R}^p \setminus \Delta.$$

Then Consider $\pi : \mathbb{R}^p \setminus \Delta \longmapsto \Delta^\perp \setminus \{0\}$ (project)

and then $r : \Delta^\perp \setminus \{0\} \longrightarrow S^{p-2}$ (radial project)



Check: <u>equivariance</u>

So $\varphi = f \circ \pi \circ f : V_2(\mathbb{R}^n) \longrightarrow S^{p-2}$

But $V_2(\mathbb{R}^n)$ is known to be $(n-3)$-connected.

$\exists$ fiber bundle $\quad \underset{V_1 \mathbb{R}^{n-1}}{\overset{\shortparallel}{\mathbb{R}P^{n-1}}} \longrightarrow V_2 \mathbb{R}^n \longrightarrow S^{n-1}$ $\quad \boxed{LES}$

& $\dim(S^{p-2}) = p-2 < n-2$.

Action of $\mathbb{Z}/p \curvearrowright S^{p-2}$ is free (need prime here)

Dold's theorem $=$ contradiction

Beautiful application:

<u>Thm (Kakutani)</u>: a frame $\{e_1, e_2, e_3\}$ is a solution to Knaster's problem for $(2,1)$, i.e.: $f : S^2 \longrightarrow \mathbb{R}$,

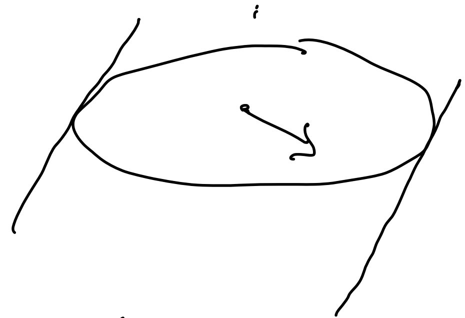
$\exists\, \rho \in SO(3) \ni f(\rho(e_1)) = f(\rho(e_2)) = f(\rho(e_3))$

convex, compact, nonempty interior

Corollary: Every convex body K in $\mathbb{R}^3$ can be inscribed in a cube

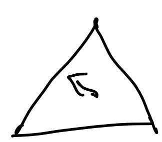
Pf: Consider $f: S^2 \to \mathbb{R}$ where $\forall v \in S^2$, $f(v)$ is the width of K in direction of v.

i.e.:

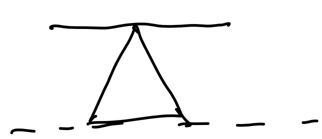


lines orthogonal

it's okay if 2-planes agree (flat object)



or



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fix a frame e_1, e_2, e_3, then $\exists$ another frame $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ (after applying orthogonal transformation) where

$$f(\tilde{e}_1) = f(\tilde{e}_2) = f(\tilde{e}_3).$$