Q1: What is the Atiyah-Hirzebruch SS?

Let $A^\bullet : \mathrm{Top}_*^{op} \longrightarrow \mathrm{Ab}$ be a reduced cohomology theory.

Define $\underline{A^q = A^q(S^0)}$.

The AHSS is a spectral sequence

$$\boxed{E_2^{p,q} = \tilde{H}^p(X, A^q) \Rightarrow A^{p+q}(X).}$$

"Universal coefficients for generalized (co)-homology"

Rmk: We can use unreduced A and H too.

Q2: How Is The AHSS obtained?

Quick Answer

Filtrations On Spaces + Nonsense

The Nonsense (Adams' Blue Book III. 7)

We use CW-filtrations, but this is all more general (Filtered objects, cofiber seq.)

$i : Y \hookrightarrow X$ gives

$$\cdots \to \underbrace{A^{n-1}(Y) \longrightarrow A^n(X/Y)} \to A^n(X) \to A^n(Y) \to \cdots$$

$$A^n\left(X/Y \simeq C_i \longrightarrow C_{X \hookrightarrow C_i} \simeq \Sigma Y \right)$$

$$\circ \; A^{n-1}(Y) \xrightarrow{\sim} A^n(\Sigma Y)$$

More on the construction

Let $X = X_0 \subseteq X_1 \subseteq \dots \subseteq X_n$.

From $X_{p-1} \hookrightarrow X_p$, we obtain

$$\cdots A^{p+q-1}(X_{p-1}) \xrightarrow{\beta} A^{p+q}(X_p/X_{p-1}) \xrightarrow{\gamma} A^{p+q}(X_p) \xrightarrow{\alpha} A^{p+q}(X_{p-1}) \to \cdots$$

Exact couple:

$$D' = \bigoplus_{p,q \in \mathbb{Z}} D^{p,q}$$

$$E' = \bigoplus_{p,q \in \mathbb{Z}} E^{p,q}$$

$$D^{p,q} = A^{p+q}(X_p)$$

$$E^{p,q} = A^{p+q}(X_p/X_{p-1}) \cong \prod_{I_p} A^q$$

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

Last One of These (I promise)

$$\boxed{E_1^{p,q} = A^{p+q}(X_p/X_{p-1}) \Rightarrow A^{p+q}(X)}$$

$$E_1^{p,q} = A^{p+q}(X_p/X_{p-1}) = \prod_{I_p} A^q$$

$$\cong H^p(X_p, X_{p-1}; A^q)$$

(cellular) $\cong C^p(X, A^q)$

$$\begin{array}{ccccc} \delta_1^{p,q} : & E'_{p,q} & \longrightarrow & E'_{p+1,q} \\ & \wr\wr & \circlearrowleft & \wr\wr \\ \delta : & C^p(X; A^q) & \longrightarrow & C^p(X; A^q) \end{array}$$

(not immediate)

$$\boxed{E_2^{p,q} \cong H^p(X; A^q)}$$

A First Computation

$$\boxed{K^i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}^{n+1} & i \text{ even} \\ 0 & \text{otherwise} \end{cases}}$$

Recall that $H^p(\mathbb{C}P^n, A) = \begin{cases} A & p \text{ even} \\ 0, & \text{odd} \end{cases}$

$\Rightarrow$ differentials are all zero, so

$$E_2^{p,q} \cong E_\infty^{p,q}.$$

Since E_∞ is torsion-free,

we get the result.

AHSS Can Get Better

"hands on"

Def: A multiplicative cohomology theory

$\mu : A^p(X,A) \otimes A^q(Y,B) \to A^{p+q}(X\times Y, X\times B \cup A\times Y)$.

(unital, associative, graded commutative)

Given: $\Delta : (X,A) \to (X\times X, A\times X \cup X\times A)$,

cup product: $\Delta^* \circ \mu$

Mult. AHSS: $(A, \mu, 1)$ multiplicative.

All the differentials in AHSS

$E_2 : H^*(B, A^*(\cdot)) \Longrightarrow A^*(X)$

are linear over A^*.

[Pages are DGA]

$\Rightarrow \quad K^*(\mathbb{C}P^n) \cong K^*[t]/t^{n+1}$

The Easy Version of $A^*(\mathbb{C}P^\infty)$

Assume A is trivial in odd degree, so $A^{2n+1}(*) = 0$ for all n. (1)

For any mult. cohomology theory satisfying (1), that

$$\boxed{A^*(\mathbb{C}P^n) = E_2(\mathbb{C}P^n) = A^*[x]/x^{n+1}.}$$

Rmk: This is true for any complex oriented cohomology A

Milnor exact sequence:

$$0 \to \underbrace{\lim^1 E^{*-1}(\mathbb{C}P^n)}_{=0} \to E^*(\mathbb{C}P^\infty) \to \underbrace{\lim_n E^*(\mathbb{C}P^n)}_{=A^*[[x]]} \to 0$$

since $E^*\mathbb{C}P^{n+1} \twoheadrightarrow E^*\mathbb{C}P^n \ \forall n$.

First Chern Class

$A^* \mathbb{C}P^\infty \cong A^*[[x]]$.

Similar arguments:

$$A^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong A^*[[x,y]],$$

where x, y are pullbacks of projection

Def: The first chern class of a line bundle L

$$c_1(L) = f^*(x)$$

with $f : X \longrightarrow \mathbb{C}P^\infty$ the classifying map.

Let $g : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$

classify tautological $\ell \otimes \ell$ so

$$g^* : E^*[[x]] \longrightarrow E^*[[x,y]].$$

$$F(x,y) = g^*(x)$$

$E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \tilde{E}^*(S^0)[[x,y]]$

More On Chern Classes

Let L_1, L_2 be line bundles on X, with classifying maps f_1, f_2 so

$$\begin{tikzcd} f^*(\mathcal{L}) \arrow[r] \arrow[d] & \mathcal{L} \arrow[d] \\ X \arrow[r, "f"] & \mathbb{C}P^\infty \end{tikzcd}$$

$$\begin{tikzcd} L_1 \otimes L_2 \arrow[r] \arrow[d] & pr_1^*(\ell) \otimes pr_2^*(\ell) \arrow[d] \\ X \arrow[r, "f_1 \times f_2"] & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \end{tikzcd}$$

$$\begin{aligned} c_1(L_1) \otimes c_1(L_2) &\cong (f_1 \times f_2)^*(c_1(pr_1^* \ell \otimes pr_2 \ell)) \\ &\cong (f_1 \times f_2)^*(F(pr_1^* \ell, pr_2^* \ell)) \\ &= F(c_1(L_1), c_1(L_2)). \end{aligned}$$

Write $x +_F y$ for $F(x,y)$

$x +_F y = y +_F x$ $\quad (L_1 \otimes L_2 \cong L_2 \otimes L_1).$

$x +_F 0 = x = 0 +_F x$ $\quad (L_1 \otimes \mathbb{1} \cong L_1 \cong \mathbb{1} \otimes L_1)$

$(x +_F y) +_F z = x +_F (y +_F z)$ $\quad$ (ass.)

$\uparrow$ Formal group

<u>Higher Chern Classes</u>

Want to use the "Grothendieck" way:

Let $p: E \longrightarrow X$ be a vector bundle and $q: \mathbb{P}(E) \longrightarrow X$ be its projectivization.

$$\begin{tikzcd}
L_E \arrow[r, hook] \arrow[dr] & q^*(E) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
& \mathbb{P}(E) \arrow[r, "q"] & X
\end{tikzcd}$$

$$L_E := \{(L, y) \in q^*(E) \mid y \in L\}.$$

$$\alpha := c_1(L_E) \in A^2(\mathbb{P}(E)).$$

<u>Claim:</u> $1, \alpha, \ldots, \alpha^{n-1}$ is a basis for $A^*(\overset{\mathbb{P}(E)_x}{\overset{\parallel}{\mathbb{CP}^{n-1}}})$

$\Rightarrow$ $A^* \mathbb{P}(E)$ is free over $A^* X$. (Leray-Hirsch)

$\Rightarrow$ $\alpha^n + q^*(c_1)\alpha^{n-1} + \cdots + q^*(c_n) = 0$

Chern classes are $c_1, \ldots, c_n$.

Final Form of AHSS.

This is called the Serre-Atiyah-Hirzebruch spectral sequence.

Let $F \to X \to B$ be a fibration with B path connected such that $\pi_1 B$ acts trivially on $A^*(F)$. Then we have a spectral sequence

$$E_2^{p,q} = H^p(B, A^q(F)) \Rightarrow A^{p+q}(X).$$

Rmks: 1) $\bullet \to X \xrightarrow{id} X$ recovers AHSS.

2) $A^* = H^*$ recovers Serre SS

~~Proof~~ Statement of Claim

Theorem: Let $F \to E \to B$ be a fiber bundle such that

1) $A^n(F)$ is finitely generated free module over A^* for all n

2) There are classes $c_j \in A^{k_j}(E)$ whose restriction form a basis for $A^*(F)$ as an A^*-module.

Then

$$\varphi : A^*(B) \otimes_{A^*} A^*(F) \xrightarrow{\sim} A^*(E)$$

$$\sum_{ij} b_i \otimes i^*(c_j) \longrightarrow \sum_{ij} p^*(b_i) \cdot c_j$$

idea ~~Proof~~ of Claim

$$A^q(E) \twoheadrightarrow E_\infty^{0,q} \subset E_{q+1}^{0,q} \subset \cdots \subset E_2^{0,q} = \overset{A^q(F)}{\overset{\parallel}{H^0(B, A^q(F))}}$$

(Assumption 2) $\quad i^* : A^*(E) \twoheadrightarrow A^*(F)$

$\Rightarrow$ all of these are equality $\Rightarrow d_r = 0$ on the q-axis

(Assumption 1) $\quad A^n(F)$ is free A^*-module.

$$E_2^{p,q} = H^p(B, A^q(F)) \cong H^p(B, A) \otimes_{A^*} H^q(F, A)$$

$\Rightarrow d_2$ is zero on $E^{p,0}$ $\quad = E_2^{p,0} \otimes E_2^{0,q}$

$\Rightarrow$ collapses on page 2.