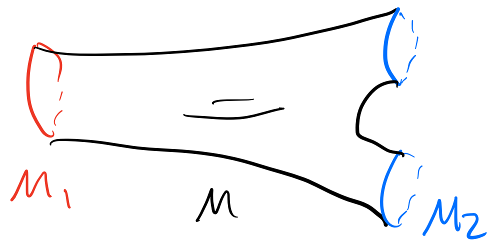
Equivariant Complex Cobordism

Outline:

- Review of Classical Cobordism
- Equivariant Cobordism & Representability
- Computations in Equivariant Cobordism
- Some Questions I have

Review of Classical unoriented Cobordism

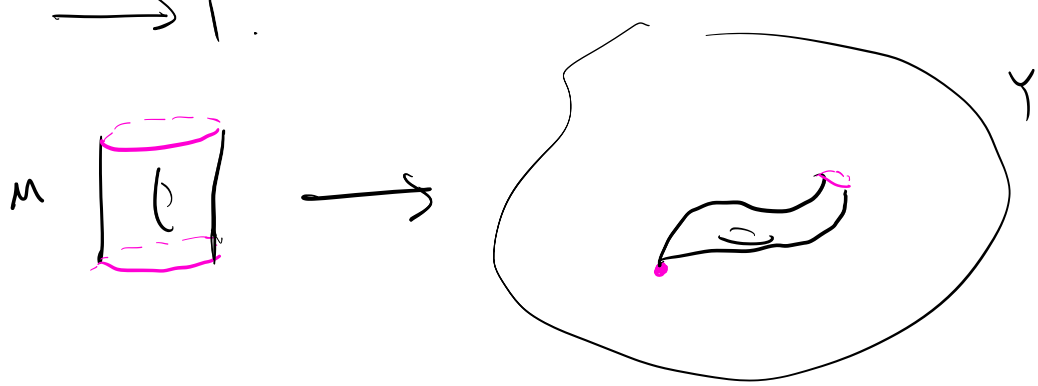


Def: two smooth n-manifolds M_1, M_2 are cobordant if there is an $(n+1)$-manifold M with $\partial M = M_1 \sqcup M_2$.

$[M^n]$ = cobordism class of n-manifolds.

$\sqcup, \times$ make this into a ring N_*.

$N_*(Y)$ cobordism classes of maps $M \longrightarrow Y$.

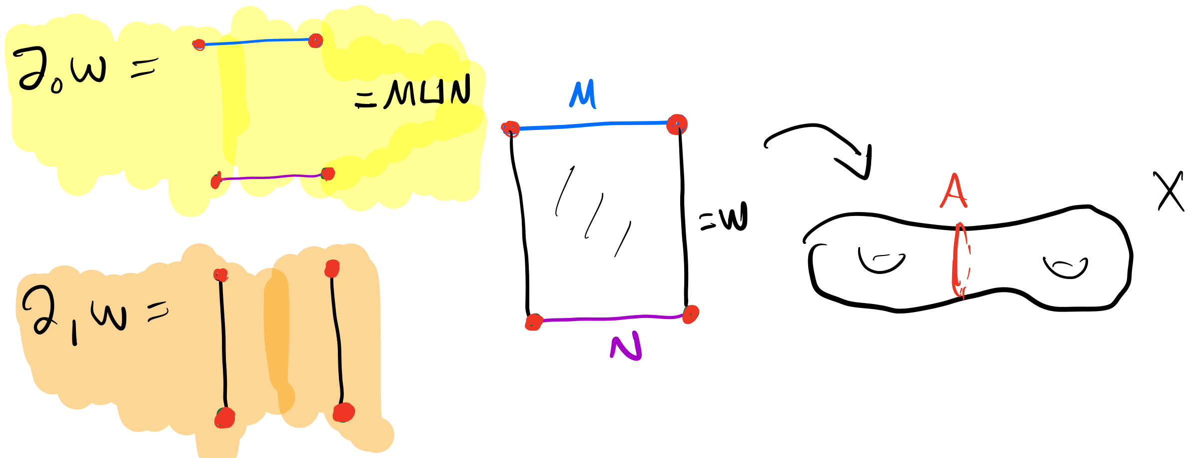


Cobordism $\Longrightarrow$ Homology

$N_*(X, A)$: Cobordism classes of maps

$(N, \partial N) \longrightarrow (X, A) \sim (M, \partial M) \longrightarrow (X, A)$

if $\exists$ $(W, \partial_0 W, \partial_1 W)$



$$\partial_0 W \cap \partial_1 W = \partial(\partial_0 W) = \partial(\partial_1 W)$$

$$\partial_0 W \cup \partial_1 W = \partial W.$$

<u>Note:</u> $N_k(X, A) \xrightarrow{\partial} N_{k-1}(A)$

is well-defined

$$\cdots \to N_k(A) \to N_k(X) \to N_k(X, A) \to N_{k-1}(A) \to \cdots$$

$\rightsquigarrow$ We have a Homology Theory.

The Representing Spectrum

Claim: $\mathcal{N}_* \cong \pi_*(MO)$ [MO represents unoriented Bordism]

$M^k \hookrightarrow \mathbb{R}^{q+k}$ w/ normal bundle ν & Thom-Space $T\nu$

Pontryagin-Thom:

$$S^{q+k} \longrightarrow T\nu \longrightarrow TO(q)$$

covers the classifying map.

Gives map $\mathcal{N}_k \longrightarrow \pi_k MO$

Conversely: $f: S^{q+k} \longrightarrow TO(q)$,

make f transverse to the zero section

of $TO(q) \longrightarrow BO(q)$.

$\Rightarrow$ $M = f^{-1}(BO(q))$ is a k-submanifold

of S^{q+k}

($\nu(M)$ is pullback of classifying map!)

Inverse of Pontryagin Thom Cont'd

$g \simeq f : S^{a+k} \longrightarrow TO(q)$

via $F : S^{a+k} \times I \longrightarrow TO(q)$

$N \simeq f^{-1}(BO(q))$

$M \simeq g^{-1}(BO(q))$

$W \simeq F^{-1}(BO(q))$

W is a cobordism, provided that we can make F transverse.

$\Rightarrow$ Map $T_* MO \longrightarrow N_*$ is well defined & an inverse to Pontryagin-Thom.

$\Rightarrow$ $\pi_*(MO) \cong N_*$ (as rings!)

<u>Remark</u>: $\pi_* MO \cong \mathbb{Z}/2\mathbb{Z}[x_n \mid n \in \mathbb{N}, n \geq 2, n \neq 2^t - 1]$.

Equivariant Cobordism & MO_G.

- Define $\mathcal{N}_*^G$ as before, but now with smooth <u>G-manifolds.</u>

Equivariant Thom Spectrum

Let $\mathcal{U}$ be a complete G-universe.

We have

$$\pi(V) : EO(|V|, V \oplus \mathcal{U}) \to BO(|V|, \mathcal{U}).$$

$TO_G(V)$ is the Thom space of $\pi(V)$.

$$V \subseteq W \Rightarrow BO(|V|, V \oplus \mathcal{U}) \to BO(|W|, W \oplus \mathcal{U})$$

Pullback is $\pi(V) \oplus \mathbb{1}_{W-V}$ w/ Thom space $\Sigma^{W-V} TO_G(V)$.

* Structure maps $\sigma : \Sigma^{W-V} TO_G(V) \to TO_G(W)$.

Failure of Representability

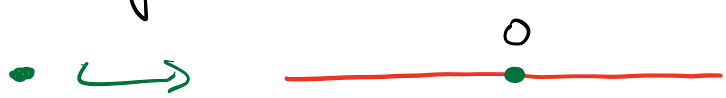
MO_G does not represent N_*^G

We have the Pontryagin-Thom map

$$N_*^G \longrightarrow \pi_*^G(MO_G).$$

Transversality is the obstruction to defining an inverse.

Ex from the book: $G = \mathbb{Z}/2$, $M = *$, $N = \mathbb{R}$, $G \circlearrowright \mathbb{R}$ by $x \mapsto -x$.

f : 

f is not transverse to $Y = \{0\}$, & you can't separate them equivariantly.

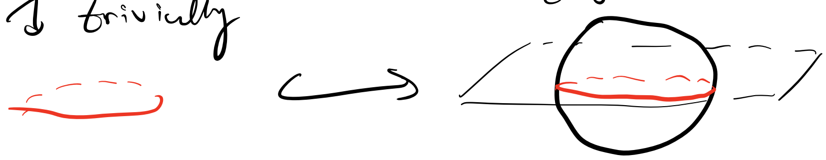
A Fix (Wasserman's Criteria)

The previous example can be generalized:

$f: M \to N$, $Y \subseteq N$,

reps in $\nu(Y)$ that aren't in the tangent bundle of M:

$\mathbb{Z}/2$ ↺ trivially



Fix: whenever we have a submanifold $Y \subseteq N$, demand reps in $\nu(Y)$, make sure that it also appears in the action on M.

(Wasserman calls this condition "consistent")

When Can We Define an inverse to the Collapse map?

Reduce nontrivial representations on universal bundles:

Replace $\mathcal{U}$ by G-fixed points, $\mathcal{U}^G \cong \mathbb{R}^\infty$.

$$EO(|V|, V \oplus \mathcal{U}^G) \longrightarrow BO(|V|, V \oplus \mathcal{U}^G).$$

Gives to_G & mo_G.

mo_G does represent $\mathcal{N}_*^G$.

The inclusion $\mathcal{U}^G \to \mathcal{U}$ induces $mo_G \longrightarrow MO_G$ that represents

$$\mathcal{N}_*^G \longrightarrow MO_G.$$

Properties of MO_G

$$MO_k^G(X,A) \cong \operatorname{colim}_V \mathcal{N}^G_{k+|V|}\big((X,A)\times (D(V),S(V))\big)$$

"Pontryagin-Thom"

Manually adding reps.

$$\Rightarrow \quad MO_k^G \cong \operatorname{colim}_V \mathcal{N}^G_{k+|V|}(D(V),S(V))$$

$$[(M,\partial M) \longrightarrow (D(V),S(V))] \in MO_k^G.$$

$$\parallel$$

$$[(M\times D(W), \partial(M\times D(W))) \longrightarrow (D(V\oplus W), S(V\oplus W))]$$

Classes of such a manifold over the disk of a rep is called a <u>stable mfld</u>

Virtual dimension is $\dim M - \dim V$.

MO_k^G is thus cobordism classes of stable mflds of dim k.

Euler Class

V a rep w/ no nontrivial summands,
$[* \hookrightarrow D(V)] = \chi(V) \in MO^G_{-n}$,
$n = |V|$.

- If V had a trivial summand, then $* \hookrightarrow D(V)$ could be homotoped to $S(V)$, and so it would vanish in $N^G_{-n}(D(V), S(V))$.

- Euler class is nontrivial (later)

* This is an element in MO^G_* that has no hope of appearing in N^G_*, which has nothing in negative dimension.

Periodicity of MO_G

- $MO_G(V) \cong MO_G(|V|)$,

so $\Sigma^V MO_G \simeq \Sigma^n MO_G$, for $n = |V|$.

Consider $V \longrightarrow *$. This gives

$$S^V \longrightarrow TO_G(\mathbb{R}^n) \hookrightarrow MO_G(\mathbb{R}^n),$$

or $$S^{V-n} = \Sigma_n^\infty S^V \longrightarrow MO_G.$$

The same for $\mathbb{R}^n \longrightarrow *$ gives $S^{n-V} \longrightarrow MO_G$.
Thus, we obtain an equivalence

$$S^{V-n} \wedge MO_G \longrightarrow MO_G \wedge MO_G \longrightarrow MO_G.$$

$$\Rightarrow \quad MO_k^G(X) \cong MO_{k+n}^G(\Sigma^V X)$$

Families & Equivariant Cobordism

Def: Let $\mathcal{F}$ be a family. An $\mathcal{F}$-manifold is a smooth G-manifold such that all isotropy groups are in $\mathcal{F}$.

Call $N_*^G[\mathcal{F}]$ to be the group of closed mflds with restricted isotropy.

For $\mathcal{F}' \subset \mathcal{F}$, we can form $N_*^G[\mathcal{F}, \mathcal{F}']$ with M an $\mathcal{F}$-manifold and ∂M an $\mathcal{F}'$-manifold.

$E\mathcal{F}$ $\Psi(\mathcal{F})$

$M \to E\mathcal{F}$ $\Rightarrow$ M is an $\mathcal{F}$-manifold | M an $\mathcal{F}$-mfld $\Rightarrow \exists!\ M \to E\mathcal{F}$

$$N_*^G[\mathcal{F}] \cong N_*^G(E\mathcal{F}),$$

$$N_*^G[\mathcal{F}](X) \cong N_*^G(X \times E\mathcal{F}).$$

$$MO_*^G[\mathcal{F}] := MO_*^G(E\mathcal{F}).$$

Nontriviality of Euler class

G compact Lie, V a rep w/o trivial summands
$\Rightarrow \chi(V) \neq 0 \in MO^G_{-n}$, $n = |V|$.

"Pf": $\mathcal{A}$ = All subgps $\mathcal{B}$ = All proper subgroups.

$$MO^G_*(E\mathcal{A}, E\mathcal{B})$$
$$\parallel$$

Take $\varphi : MO^G_* \longrightarrow MO^G_*[\mathcal{A}, \mathcal{B}]$.

claim $\varphi(\chi(V))$ is invertible $\Rightarrow \chi(V) \neq 0$.

Recall that $\chi(V) = [* \hookrightarrow D(V)] \in MO^G_*$.

$\varphi(\chi(V))^{-1} = [D(V) \to *] \in MO^G_*[\mathcal{A}, \mathcal{B}]$,
since $\partial D(V) = S(V)$ has no fixed points.

Spectral Sequence & Induction

$$\cdots \to N_k^G[\mathcal{F}'] \to N_k^G[\mathcal{F}] \to N_k^G[\mathcal{F},\mathcal{F}'] \xrightarrow{\partial} N_{k-1}^G[\mathcal{F}] \to \cdots$$

Choose a filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ of all subgroups whose union is the family of all subgroups.

- Inductively understand $N_k^G[\mathcal{F}_0]$ & $N_k^G[\mathcal{F}_p, \mathcal{F}_{p-1}]$.

Exact couple:

$$\begin{array}{ccc} N_*^G[\mathcal{F}_{p-1}] & \longrightarrow & N_*^G[\mathcal{F}_p] \\ & \nwarrow \quad \swarrow & \\ & N_*^G[\mathcal{F}_p, \mathcal{F}_{p-1}] & \end{array}$$

$$E^1_{p,q} = N_q^G[\mathcal{F}_p, \mathcal{F}_{p-1}] \Rightarrow N_*^G$$

Page 1 : $N_q^G[\mathcal{F}_p, \mathcal{F}_{p-1}]$

$$N_*^G[\{e\}, \emptyset] = N_*^G[\{e\}].$$

equivariant Bordism of free closed G-manifolds

M/G is also a manifold of $\dim M - \dim G$, classifying map $M/G \to BG$ that respects cobordism relation, so

$$N_*^G[\{e\}] = N_{*-\dim G}(BG).$$

★ ↑ nonequivariant

<u>Goal</u>: understand $N_q^G[\mathcal{F}, \mathcal{F}']$ by looking at "adjacent families", so $\mathcal{F} = \mathcal{F}' \cup (H)$, where (H) is the conjugacy class of H.

(Reduce to a nonequivariant cobordism ring)

$\mathcal{N}_q[\mathcal{F}_p, \mathcal{F}_{p-1}]$ cont.d

$\mathcal{F} = \mathcal{F}' \cup (H)$, G finite.

Let $M^{(H)}$ be the subset of M w/ isotropy groups in H.

* $M^{(H)} \subset \text{int}(M)$, since ∂M is an $\mathcal{F}'$-manifold
  & $M^{(H)} = \bigcup_{K \in (H)} M^K$ is a union of closed mflds.

Moreover M^K are all disjoint & in fact
$$M^{(H)} \cong G \times_{NH} M^H.$$

Let N be a closed tubular neighborhood of $M^{(H)}$

$\Rightarrow (M, \partial M)$ is cobordant to $(N, \partial N)$
& $(N, \partial N)$ is determined by by free WH-manifold M^H & NH-bundle $\nu(M^H)$.

* decompose bundle by irr. representations in each fiber
  (Q: why does this extend to the whole bundle?)

$N_q[F_p, F_{p-1}]$ cont.d

Let $(V_1, \dots V_m)$ be the irreducible representations of H.

$\Rightarrow \nu(M^H) = \bigoplus \nu_i(M^H)$, where

$\nu_i = \bigoplus_k V_i$.

But ν_i is determined by the free WH-bundle $\mathrm{Hom}_G(V_i, \nu_i)$ with fibers $\mathbb{R}^n$

The point: $[M] \in N_k^G[\mathcal{F}, \mathcal{F}']$ can be thought of as free WH-manifolds together w/ a sequence of WH-bundles $\alpha_1, \dots, \alpha_m$ over M w/ structure group $O(n_i)$

Use this to see that

Non-equivariant!

$$N_k^G[\mathcal{F}, \mathcal{F}'] \cong \sum_{\substack{\dim WH + j + \\ \sum n_i d_i = k}} N_j\left(EWH \times_{WH} \left(\prod_i BO(\mathbb{R}, n_i)\right)\right)$$

Some Questions:

Global sp. lines
eq. cob. rings

1) MU_G^* classifies equivariant FGLs. Equivariant Complex orientation $\Rightarrow$ Equivariant FGL & likewise there is a Conner-Floyd isomorphism $\widetilde{MU}_G^*(X) \otimes_{MU_G^*} K_G^* \to \tilde{K}_G^*(X)$.

Any Landweber exactness theorem?

Wolfgang Lück
Transformation

2) There is an s-cobordism theorem for G-manifolds classifying geometric cobordism (for manifolds satisfying the "weak gap" hypothesis) Can one use this to prove an equivariant "generalized Poincaré" conjecture?

$\surd$?

(G-transversality revisited)

3) If we restrict to "normal" cobordism, the obstruction to stable G-transversality for $f: M/\partial M \to T(V)$ over Y arises as a class in the cofiber of $N_*^G(M, \partial M; Y, V) \to MO_*^G(M, \partial M; Y, V)$

Does every class in the cofiber arise this way?

Problems

1) $\mathbb{Z}/3 \curvearrowright S^2$ by rotation $\mathcal{F}' = \{e\}$, $H = \mathbb{Z}/3$.

Show that $(M, \partial M)$ is cobordant to $(N, \partial N)$ with N a tubular neighborhood of $S^{2\,(H)}$

N is $S^2 \times I$ with smoothed corners.

* (Show this fact more generally)

2) Why can we extend a decomposition of $\nu(n)|_p$ into irr. representations to the entire base?

3) Show that

$$MO_k^G \cong \operatorname{Colim}_V \mathcal{N}^G_{k+|V|}(D(V), S(V))$$

Using Wasserman's criteria as in "G-transversality Revisited" Prop 2.5

4) Show that $\chi(V)$ really has an inverse in $MO_*^G[\mathcal{A}, \mathcal{B}$

5) Isotropy separation + Lewis-May-Steinberger gives a diagram

$$\begin{array}{ccccc} (E\mathbb{Z}/p \wedge MU_{\mathbb{Z}/p})^{\mathbb{Z}/p} & \longrightarrow & (MU_{\mathbb{Z}/p})^{\mathbb{Z}/p} & \longrightarrow & \Phi^{\mathbb{Z}/p} MU_{\mathbb{Z}/p} \\ \downarrow & & \downarrow & & \downarrow \\ (E\mathbb{Z}/p_+ \wedge F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/p}))^{\mathbb{Z}/p} & \longrightarrow & F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/p})^{\mathbb{Z}/p} & \longrightarrow & MU_{\mathbb{Z}/p}^{t\mathbb{Z}/p} \end{array}$$

How does this imply that we have a pullback of rings

$$\begin{array}{ccc} (MU_{\mathbb{Z}/p})_* & \longrightarrow & MU_*[u_k, u_k^{-1}, b_k^{(i)} \mid i > 0, k \in \mathbb{Z}/p^\times] \\ \downarrow & & \downarrow \\ MU_*[[u]]/([p]_f u) & \longrightarrow & MU_*[[u]]/[p]_f u[u^{-1}] \end{array}$$

https://www.maths.ed.ac.uk/~v1ranick/papers/costwan2.pdf

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