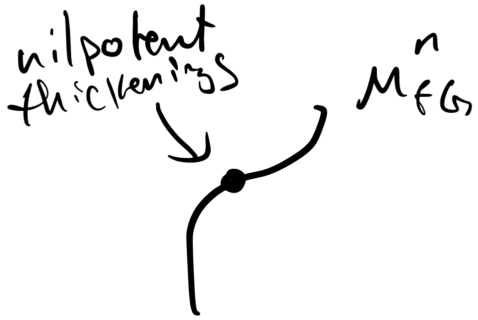
Lubin-Tate Theory

Goal: study what happens in the nbhd of a point in $\mathcal{M}_{FG}^n$.

i.e. Deformations of a FGL.

Fix a perfect field k, $F(x,y) \in k[[x,y]]$.



$T \subset T'$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \tilde{Y} \arrow[d] \\ X \times I \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d] \\ T' \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

Def: A Infinitesimal thickening of k is a ring $\varphi: A \to k$,

$$m_A = \ker \varphi$$

(1) $m_A^a = 0 \quad a \gg 0.$

(2) m_A^a / m_A^{a+1} f.d. vector spaces

Def: (Deformation) A deformation of F over A (infinitesimal thickening) is a FGL F_A over A

$$\varphi^* : FGL(A) \longrightarrow FGL(k)$$

$$\varphi^*(F_A) = F.$$

$f \in A[[t]]$ such that $f(t) \equiv f \mod m_A$

$\mathrm{Def}_k(A)$ = isomorphism classes of deformations of F over A.

* not assume that k is perfect

<u>Thm (Lubin-Tate)</u>: Let F be the FGL, $\mathrm{ht}\, F = n < \infty$ over k,

Then $\mathrm{Def}(A)$ is a discrete groupoid with

$$\pi_0(\mathrm{Def}_k(A)) \simeq m^{\times n-1}$$

&

Moreover, $\exists$ a complete local ring $E(k, F)$ with an isomorphism

$$i : k \xrightarrow{\sim} E(k, F)/m$$

& a FGL G on $E(k, F)$

such that G is a universal deformation

$\{ \phi : E(G, F) \to A \} \Leftrightarrow \mathrm{Def}\, A$

$$\phi^*(G(x,y)) = \sum_{i,j} \phi(a_{ij}) x^i y^j$$

When K is a perfect field, we can take

$$R = W(K)[[v_1, \ldots, v_{n-1}]]$$

$R \to K$, kernel $(p, v_1, \ldots, v_{n-1})$

$$\varphi_0 : L_{(p)} \longrightarrow K$$

$$\parallel$$

$$\mathbb{Z}_{(p)}[t_1, \ldots]$$

$$\varphi_0 : L_{(p)} \longrightarrow k$$
$$\qquad \shortparallel$$
$$\mathbb{Z}_{(p)}[t_1, \dots]$$

$$\varphi_0(t_{p^i-1}) \mapsto 0 \quad \text{for all}$$
$1 \leq i \leq n-1$

$\varphi : L_{(p)} \longrightarrow R$ be any
homomorphism lifting φ_0.
& taking $t_{p^i-1} \mapsto v_i$.

This defines $\tilde{F} \in FGL(R)$

<u>Thm:</u> $(W(k)[[v_1, \dots, v_{n-1}]], \tilde{F})$ is
the universal deformation for F.

when k is perfect,

$$\begin{tikzcd} Wk \arrow[r, dashed, "\exists!"] \arrow[d] & B \arrow[d] \\ k \arrow[r] & B/m \end{tikzcd}$$

B is a complete local ring

$\Rightarrow$ $E(k, F)$ a Wk-algebra.

$Wk \subseteq E(k, F)$

Any deformation determines a ring map

$Wk \to B$.

$k = \mathbb{F}_p \quad \rightsquigarrow W(k) = \mathbb{Z}_p$

<u>Punchline</u> The universal deformation $\tilde{F}$ over $R = W(k)[[v_1, \ldots, v_{n-1}]]$ is Landweber exact, since $p, v_1, \ldots, v_{n-1}$ is a regular

(LEFT) $\Rightarrow$ $E(n)$ & this is the nth Morava E-theory

Moreover, v_n is invertible in $R/(v_0, \ldots, v_{n-1}) \cong k$,

(Lurie lecture 18) $E(n)$ is even periodic &

$$\pi_*(E(n)) \cong W(k)[[v_1 \ldots v_{n-1}]][\beta^{\pm 1}]$$

$$|\beta| = 2.$$

$L_{E(n)}$ "restriction" to $\mathcal{M}_{FG}^{\leq n}$ (Lecture 22)

Morava-Stabilizer Group (Lecture 19)

Let $k = \overline{\mathbb{F}_p}$ & let F be the unique FGL of height n.

$\operatorname{Spec} \overline{\mathbb{F}_p} \to \mathcal{M}_{FG}^n$ is faithfully flat

$\forall R$ & FGL G on R, $\exists$ a pullback

$$\begin{tikzcd} \operatorname{Spec}(R') \arrow[r] \arrow[d] & \operatorname{Spec}(R) \arrow[d] \\ \operatorname{Spec} \overline{\mathbb{F}_p} \arrow[r] & \mathcal{M}_{FG}^n \end{tikzcd}$$

where R' is a direct limit of finite étale extensions of $R \otimes \overline{\mathbb{F}_p}$ (faithfully flat)

Take $B = \overline{\mathbb{F}_p}$

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} \overline{\mathbb{F}_p} \arrow[d] \\
\operatorname{Spec} \overline{\mathbb{F}_p} \arrow[r] & \mathcal{M}^n_{FG}
\end{tikzcd}$$

Spec B is a direct limit of finite étale extensions.

B as a topological space is an inverse limit of finite sets

$$\cdots \to X_1 \to X_0$$

we denote this by G.

$$\{ * \hookrightarrow G \} = \{ B \to k \} / \text{iso}$$

where k is some alg. closure of $\mathbb{F}_p$.

<u>i.e.</u>:

(1) $n, n' : \overline{\mathbb{F}_p} \to k$

(2) iso $\quad n^*(F) \cong (n')^* F$ over k.

$n' = id$, $k = \overline{\mathbb{F}_p}$

$$n \in \mathrm{Aut}(\overline{\mathbb{F}_p}) = \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) = \hat{\mathbb{Z}}$$

& $\quad n^* F \cong F$.

$\mathbb{G} \cong \mathrm{Aut}(\overline{\mathbb{F}}_p, F)$

$0 \to \mathrm{Aut}(F) \to \mathrm{Aut}(\overline{\mathbb{F}}_p, F) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 0$

$\Rightarrow \mathbb{G}$ the Morava stabilizer group

Lurie lecture 19

$M^n_{FG} \simeq \mathrm{Spec}\, \overline{\mathbb{F}}_p // \mathbb{G}$,

$\mathbb{G} \to \mathrm{Gal}(\overline{\mathbb{F}}_p, \mathbb{F}_p)$.

$$\mathrm{Def}_k A \cong \mathrm{Hom}(W(k)[[v_1, \ldots, v_{n-1}]], A)$$

$\mathbb{G} \curvearrowright \mathrm{Def}_k A$.

i.e.: let $g \in \mathrm{Aut}(F)$.
choose a lift $\tilde{g}(x) \in A[[x]]$,
then for any $G \in FGL(A)$ lifting F.

$$\tilde{F}(x,y) = \tilde{g}^{-1} G(\tilde{g}(x), \tilde{g}(y))$$

$\mathbb{G} \curvearrowright W(k)[[v_1, \ldots, v_{n-1}]]$ "$FGL(A) \to FGL(A[m])$"

Lurie lecture 22.

1) Understand $E(n)$, $L_{E(n)}$

2) Understand the action of the Morava stabilizer group on $E(n)$

Rezk Hopkins-Miller Theorem