Obstruction Theory

Goal: Show how different ideas in homotopy come from obstruct..

- Representability of cohomology
- Characteristic Classes

*1) Postnikov Towers *2) Quillen +-construction

Basic Motivation: $A \subset X$,

$f: A \to Y$. Can we extend this to

$f: X \to Y$?

- When f is cellular (X, Y CW-complexes) we can try induction.

- We will assume that Y is simple. This let's us add different elements to $\pi_n(Y, y)$ w/ different basepoints. [force $[S^n, Y] = \pi_n Y$.]

Start w/ $A = X^{(0)}$

Suppose that we are given a map

$f: A \cup X^{(n)} \to Y$.

Extend it to $A \cup X^{(n+1)}$.

Each (oriented) n-cell e^{n+1}, we have the attaching map

$$S^n \xrightarrow{\alpha^n} X^{(n)} \xrightarrow{f} Y.$$

In other words, we have a map

$$C_{n+1}(Y) \longrightarrow \pi_n(Y)$$

by extending linearly

an assigment to

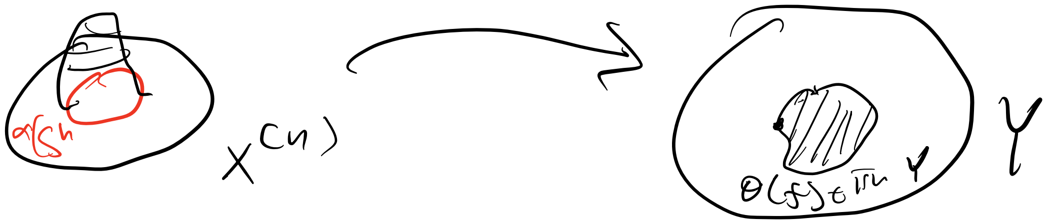
$$\theta(f) \in C^{n+1}(X, \pi_n Y).$$

<u>Key Points</u>:

1: A map is homotopically trivial in $Y \iff$ it extends to a map of D^n

i.e: Extends f to the

n-cell of X.



2) $\theta(f)$ is a cocycle, i.e. $\delta(f) = 0$.

Hurewicz Theorem (omit the proof)

pf:

$$\begin{tikzcd}
\pi_{n+2}(X_{n+2}, X_{n+1}) \arrow[r, "\varphi"] \arrow[d] & H_{n+2}(X_{n+2}, X_{n+1}) \arrow[d] \arrow[ddd, bend right, "\delta\theta(f)"'] \\
\pi_{n+1}(X_{n+1}) \arrow[r] \arrow[d] \arrow[dd, bend right, "{=0 \text{ since we have LES of homotopy}}"'] & H_{n+1}(X_{n+1}) \arrow[d] \\
\pi_{n+1}(X_{n+1}, X_n) \arrow[r] \arrow[d] & H_{n+1}(X_{n+1}, X_n) \arrow[d, "\theta(f)"] \\
\pi_n(X_n) \arrow[r] & \pi_n Y
\end{tikzcd}$$

Plan of attack to use $\theta(f)$:

want to show that

f extends $\Leftrightarrow$ $\theta(f)$ is <u>cohomologous</u> to zero.

Idea: restrict $f|_{X^{(n-1)}}$, deform it & then extend.

(one step backward, two steps forward)

- Cochain is obstruction to extending f to $X^{(n+1)}$
- <u>Cohomology class</u> is the obstruction to extending $f|_{X^{(n-1)}}$ to X^{n+1}

<u>Big Thm:</u> $\theta^{n+1}(f)$ is cohomologous to zero, then $f|_{X^{(n-1)}} : X^{(n-1)} \to P$ extends over the $(n+1)$-skeleton X_{n+1}

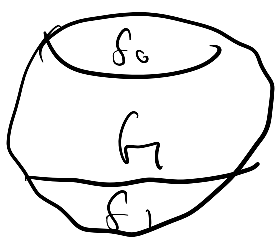
Goal #1: $f_0, f_1 : X_n \to Y$ w/ $f_0|_{X_{n-1}} \simeq f_1|_{X_{n-1}}$

Then a choice of homotopy is

$d \in C^n(X, \pi_n Y)$ w/ $\delta d = \theta(f_0) - \theta(f_1)$

Write $S^n = \partial(D^n \times I)$ for an n-cell e_i^n,

$e_i : (D^n, S^{n-1}) \to (X_n, X_{n-1})$ be char map.



$f_0 \cup g \cup f_1 \circ e_i^n \times I$

<u>Corollary:</u> $g : X_n \to Y$, $\theta^{n+1}(g)$ is null-homologous

$\iff g|_{X_{n-1}} : X_{n-1} \to Y \simeq$ a map extending

over X_{n+1}

<u>Want:</u> $\forall\ \delta d \in C^n(X, \pi_n Y)$, any map g & homotopy G, $\exists\ f_1 : X^{(n)} \to Y$ w/ $G(-,1) = f_1|_{X^{(n-1)}}$

$d \in C^n(X, \pi_n Y)$ w/ $\delta d = \theta(f_0) - \theta(f_1)$

Now apply to homotopy $G : (x,t) \mapsto f(x)$ from $g|_{X_{n-1}}$ to itself.

Let $f : X^{(n)} \to Y$ be such that $\theta f = \delta d$.

& set $f' = G(-,1)$ (we got this map)

Then

$$\delta d = \theta(f) - \theta(g').$$

$$\Rightarrow \theta(g') = 0$$

$\Rightarrow g'$ extends to X^{n+1}.

$A \subset X$, $f : A \cup (X^n) \to Y$ extends to $f : A \cup (X_{n+1}) \iff \theta(f) \in H^{n+1}(X, A, \pi_n Y)$ vanishes.

Instant applications: (X, A)

$\dim (X/A) = n$, Y $(n-1)$-connected.

Any map $A \to Y$ extends since
$H^{i+1}(X, A, \pi_i Y) = 0 \quad \forall i.$

$\Rightarrow$ Next interesting, $H^{k+1}(X, A, \pi_k Y) \neq 0$ for only one k. $O(f)$ is primary obstruction to extending to $k+1$.

$\star$ These are invariant under homotopy

$K(G, n)$ Spaces

Let X be a 2-complex with 1 zero cell., $\pi_1 X = G$.

Let Y be any space.

By cellular approx: may ... cell assume Y is a 2-complex [illegible] as well [illegible]

Given $f: X^{(1)} \longrightarrow Y$, extending f to $X^{(2)} = X \iff$ asking if a function

$$F: \pi_1 X \longrightarrow \pi_1 Y \text{ is a group homomorphism}$$

$f: X^{(1)} \longrightarrow Y^{(1)}$ is a map of generators to generators.

Given a homomorphism

$\varphi: \pi_1 X \longrightarrow \pi_1 Y$ Abelian., are there conditions to ensure that $\exists\ f: X \to Y$ realizing this homomorphism? Yes:

if $\pi_i(Y) = 0 \ \forall\ i > 1$, then we can write a map

$f: X^{(2)} \longrightarrow Y$ & then call obstructions lives in $C^{n+1}(X, \pi_n Y) = 0$

$\forall\, n > 1$.

Such spaces are called $K(G,1)$-spaces,
where $G = \pi_1 Y$

$$
\begin{tikzcd}
{[X, K(G,1)]} \arrow[r, "\pi_1"] \arrow[rdd, bend right, "\text{primary obstruction}"'] & \mathrm{Hom}(\pi_1 X, G) \\
& \cong \mathrm{Hom}(H_1 X, G) \\
& \cong H^1(X, G).
\end{tikzcd}
$$

<u>Injectivity:</u> $f : X \to K(G,1)$.

If $\theta(f) - \theta(\mathrm{const}) = 0 \Rightarrow f \simeq *$ on the n-skeleton, but obstruction theory $\Rightarrow$ all higher obstructions vanish, so we can extend this nullhomotopy to all of X.

$\Rightarrow$ if $\theta(f) = 0 \Rightarrow f \simeq *$.

(we need more work)

$$[X, K(G,1)] \cong H^1(X;G).$$

ex: $K(\mathbb{Z},1) = S^1$, $K(\mathbb{Z}/2) = \mathbb{R}P^\infty$.

$K(G,n) = \pi_n(Y) = G$, $\pi_i(Y) = 0 \ \forall i \neq n$.

Similar work shows:

$$[X, K(G,n)] \cong H^n(X;G).$$

Characteristic Classes

For many geometric applications we need a "parametrized version" of the obstruction cocycle.

Suppose that we have

$F \to E \xrightarrow{p} X$ a fiber bundle (fibration)

(F simple!)

Q: An obstruction cocycle for sections $s : X \to E$? Yes.

<u>Thm</u>: Given $s : X^{(i)} \to E$ $\exists$ a cellular cochain $\theta(s) : C_{i+1}X \to E$ that vanishes iff s extends to X^{i+1}. $\theta(s)$ is a cocycle & its cohomology class vanishes $\iff$ $s|_{X^{i-1}}$ extends to X^{i+1}.

If $\pi_k F = 0$ for $k < i$, $\theta(s)$ doesn't depend on choice of section, so $\theta_i(p) : C_{i+1}X \to \pi_i F$ is well-defined.

Start w/

$$\varphi : (D^{i+1}, S^i) \longrightarrow (X^{i+1}, X^i)$$

$$\begin{tikzcd}
 & & F \arrow[d, hook] \\
 & & E \arrow[d, bend right, "\pi"'] \\
S^i \arrow[r, hook] & (D^{i+1}, S^i) \arrow[r, "g_t"', "\varphi"] \arrow[ur, "\tilde{g}_0 = s\varphi"] & (X^{i+1}, X^i) \arrow[u, bend right, "s"']
\end{tikzcd}$$

<u>Note:</u> $\varphi|_{S^i}$ is null homotopic in X^{i+1} defined by $g_t = \varphi|_{(1-t)S^i}$

Lift this homotopy w/ $\tilde{g}_0$.
We get a lift to a homotopy

$$\tilde{g}_0 \simeq \tilde{g}_1 : S^i \longrightarrow \pi^{-1}\{*\} = F.$$

$$\theta(\varphi) = \tilde{g}_1 \in \pi_i F$$

[illegible]

Application (Hairy Ball Thm):

Suppose we have a nonvanishing vector field on S^2, i.e:

section $s : S^2 \longrightarrow TS^2$.

$v \mapsto \frac{v}{\|v\|}$ defines section

$s : S^2 \longrightarrow US^2$.

Write $X = S^2 = e^0 \cup e^2$. Define a section

$e^0 \longrightarrow US^2$ (pick a unit tangent vect at southpole)

$= X^{(1)} \longrightarrow US^2$.

$s\varphi : (D^2, S^1) \longrightarrow US^2$.

A homotopy of $\partial S^1 \to *$ in D^2 is a homotopy between constant loops @

$N, S_0 \xrightarrow{g} S^2$

lift of this homotopy:

lifting this homotopy to agree w/ $s\varphi$ at $t=0$.

multiplies generator of $\pi_1 S^1$ by two at $t=1$,

$\Rightarrow \quad \theta(S) \neq 0 \Rightarrow$

On to Stiefel-Whitney classes

$\underline{w_1}$: $L \to X$, $\pi_1(X) \longrightarrow \mathbb{Z}/2$

$$\begin{aligned} &\pi_1(X) \longrightarrow \mathbb{Z}/2 \\ &\quad \| \\ &H_1 X \to \mathbb{Z}/2 \\ &\quad \| \\ &H^1(X, \mathbb{Z}/2). \end{aligned}$$

w_1 obstructs orientability.

Thm: If the i^{th} Stiefel Whitney class $w_i E$ of a V.B over a CW complex is nonzero, $\Rightarrow \nexists$ $n-i+1$ lin ind sections.

ex: $w_n \neq 0 \Rightarrow$ every section vanishes

k linearly ind. sections is a section of Stiefel manifold

$$V_k \mathbb{R}^n \to V_k E \to B.$$

$\mathbb{Z}/2, \mathbb{Z} \cong \pi_{n-k} V_k \mathbb{R}^n$ is first nonvanishing homotopy group.

$$\Rightarrow \quad \theta_{n-k+1} V_k E \in H^{n-k+1}(X, \pi_{n-k} V_k \mathbb{R}^n)$$

Vanishes $\Leftrightarrow \exists$ a section of $V_k E$ on the $(n-k+1)$-skeleton of X.

Define $\omega_{n-k+1} := \theta_{n-k+1} V_k E$

(possibly reducing mod 2)