<u>Goal:</u> <u>Compute $\pi_*(MU)$</u>.

Recall that MU is complex oriented, so there is a universal map $\varphi : L \longrightarrow \pi_* MU$.

WTS: φ is an isomorphism.

<u>Method:</u>

1) Compute $H_*(MU)$

2)
$$\begin{tikzcd} L \arrow[r] \arrow[rr, bend right, "\text{rational equiv}"'] & \pi_*(MU) \arrow[r] & H_*(MU) \end{tikzcd}$$

Today

3) Now $\pi_*(MU)$ is f.g abelian group. It will suffice to see

$$L \longrightarrow \pi_*(MU)$$

is a an iso after p-adic completion at all primes (Adams Spectral Sequence.)

Recollections from Tommy pt. II

Let E be a complex oriented spectrum.

- $MU(1)$ is the desuspension of the Thom space for the tautological bundle $\mathcal{L} \to \mathbb{C}P^\infty$, but this is homotopy equivalent to $\Sigma^{-1}\mathbb{C}P^\infty$
- Thom iso:

$$E^*(MU(1)) \cong E^{*-1}(\mathbb{C}P^\infty)$$

$$\text{“}=\text{''}\ t \cdot E_*[[t]] \subseteq E_*[[t]]$$

$$\cong E_*(\mathbb{C}P^\infty)$$

Homology of MU: Prelude

There is a non-degenerate pairing in AHSS

$$H^*(\mathbb{C}P^n, E_*) \Rightarrow E^*(\mathbb{C}P^n)$$

$$\times$$

$$H_*(\mathbb{C}P^n, E_*) \Rightarrow E_*(\mathbb{C}P^n)$$

$\Rightarrow$ $E_*(\mathbb{C}P^\infty)$ is a free E_*-module on $\beta_0, \beta_1, \dots$ dual to $1, t, t^2, \dots$

(Since Page 2 collapses w/ zero differentials)

Thom-Isomorphism implies more generally:

$$E_*(MU(n)) = \mathrm{Sym}^n(E_* MU(1))$$

$$E_*(MU) \simeq \operatorname{colim} E_* MU(n) \cong \operatorname{colim}_i \Big(\underset{\substack{\| \wr \\ E_{*+i}(X_i)}}{E_{*+i+1}(\Sigma X_i)} \to E_{*+i+1}(X_{i+1}) \Big)$$

($\mathrm{Map}(\mathrm{Ho\,colim}\, MU(n), E) \cong \mathrm{Holim}\, \mathrm{Map}(MU(n), E)$)

+ Milnor exact sequence

More on $E_*(MU)$

what is

$$E_*(MU(n)) \longrightarrow E_*(MU(n+1))?$$

$$MU(n) \simeq MU(0) \wedge M(n) \hookrightarrow MU(1) \wedge MU(n) \to MU(n+1)$$

for $n=0$: $MU(0) \longrightarrow MU(1)$ induces

our choice of b_0.

In general

$$\begin{tikzcd}
\mathrm{Sym}^n E_*\{b_0, \dots\} \arrow[r, "\times b_0"] \arrow[d, "\wr"'] & \mathrm{Sym}^{n+1} E_*\{b_0, \dots\} \arrow[d, "\wr"] \\
E_*(MU(n)) \arrow[r] & E_*(MU(n+1))
\end{tikzcd}$$

Lurie Prop 4.7 (or BU computation) $\Rightarrow$

$$E_*(MU) = E_*[\beta_1, \beta_2, \dots]$$

Recollections From Elijah 1:

$$H_* MU = \mathbb{Z}[\beta_1, \ldots]$$

Recognize this ring

Recall that over $\mathbb{R}$

$$g(x) = x + b_1 x^2 + b_2 x^3 + \cdots$$

$$g(g^{-1}(x) + g^{-1}(y))$$ is a FGL over $\mathbb{Z}[b_1, b_2, \ldots]$

$\Rightarrow$ Characteristic map $\varphi : L \longrightarrow \mathbb{Z}[b_1, \ldots]$ is a rational isomorphism

(Lurie Lecture 2: Prop #10)

How To Conclude

We have $L \xrightarrow{\varphi} \pi_* MU \xrightarrow{H} H_* MU = \mathbb{Z}[\beta_1, \ldots]$

corresponds to a FGL. If that law

is $g(g^{-1}(x) + g^{-1}(y))$ with

$$g(x) = x + \beta_1 x^2 + \beta_2 x^3 + \cdots$$

$\Rightarrow H \circ \varphi$ is a rational isomorphism.

$\Rightarrow \varphi$ is a rational isomorphism.

☆ Our goal now is to identify the FGL determined by $H \circ \varphi$.

FGLs On $E_* MU$

Let E be any complex oriented cohomology.

$MU \wedge E$ has two complex orientations.

One from MU, and one from E. &

$\pi_*(MU \wedge E) = E_* MU = \pi_* E[b_1, \ldots]$

i.e: $x_E, x_{MU} \in \widetilde{MU \wedge E}(\mathbb{CP}^\infty)$

$$\Rightarrow (\pi_* E)[\beta_1, \ldots][[x_E]] \simeq (MU \wedge E)^*(\mathbb{CP}^\infty)$$

$$\simeq \pi_* E[\beta_1, \ldots,][[x_{MU}]].$$

$$\Rightarrow x_{MU} = \sum_{i \geq 1} a_i x_E^{i+1}$$

for $a_i \in \pi_* E[\beta_1, \ldots]$

(Lurie 7.4) $a_i = \beta_i$, $x_{MU} = x_E + b_1 x_E^2 + \cdots$

Recall: Complex Orientation $\Rightarrow$ FGL

$\pi_1, \pi_2 : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$,

$m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$ (classify $\mathcal{L}_1 \otimes \mathcal{L}_2$)

$$m^*(\gamma_E) = f_E(\pi_1^* \gamma_E, \pi_2^* \gamma_E)$$

$$m^*(\gamma_{MU}) = f_{MU}(\pi_1^* \gamma_{MU}, \pi_2^* \gamma_{MU}).$$

Substituting $\gamma_{MU} = \sum_{i \geq 1} \beta_i \gamma_E^{i+1}$

$= g(\gamma_E)$,

$$f_{MU}(x,y) = g \circ f_E(g^{-1}x, g^{-1}y)$$

$$\boxed{\begin{array}{l}\text{when } E = H\mathbb{Z}, \\ f_{MU}(x,y) = g(g^{-1}x + g^{-1}y)\ !\end{array}}$$

Finale

$$\Rightarrow \quad \begin{tikzcd} L \arrow[r] \arrow[rr, bend right] & \pi_* MU \arrow[r] & \pi_*(MU \wedge H\mathbb{Z}) \end{tikzcd}$$

is the map $\varphi : L \to \mathbb{Z}[\beta_1, \dots]$

classifying the FGL

$$g(g^{-1}x + g^{-1}y)$$

and is hence a rational isomorphism

Addendum (Just in Case):

Hurewicz map:

$$\begin{tikzcd} L \arrow[r] \arrow[rr, bend right] & \pi_*(MU \wedge \mathbb{S}) \arrow[r] & \pi_*(MU \wedge H\mathbb{Z}) \end{tikzcd}$$

corresponds to FGL from MU, f_{MU}

$$\begin{tikzcd} L \arrow[r] \arrow[rr, bend right, "f_{H\mathbb{Z}}"'] & \pi_*(\mathbb{S} \wedge H\mathbb{Z}) \arrow[r] & \pi_*(MU \wedge H\mathbb{Z}) \end{tikzcd}$$