Thom Spectra

Outline

- Thom Construction & Spectrum
  - * Vector Bundles ⇒ Thom Space
  - * Constructing The Thom Spectrum
  - * Some Properties of the Thom Spectrum

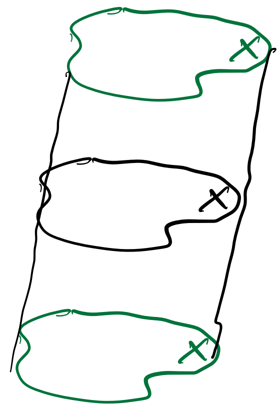
- Cobordism & The Pontryagin-Thom Construction
  - Review of (unoriented) cobordism
  - The collapse map & $\pi_* MO$.
  -

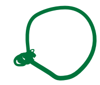
- Orientations in Cohomology
  - * E-orientations on vector bundles
  - * Orientations on cohomology theories

(Ahistorical?) Motivation for the Thom Space

Reduced Suspension:

$\Sigma X = S^1 \wedge X_+ \;=\; S^1 \times X_+ \Big/ \begin{matrix} S^1 \times \{x_0\} \\ \{0\} \times X \end{matrix}$





Different Perspective: $X \times \mathbb{R}^1$, fiberwise 1 point compactification to get $S^1 \times X$, then identify the remaining copy $\{0\} \times X$. This is $Th(\underline{\mathbb{R}})$

More generally: $\Sigma^n X \cong Th(\underline{\mathbb{R}}^n) = Th(X \times \mathbb{R}^n)$.

Goal #1: Extend this construction for twisted products with $\mathbb{R}^n$, i.e: Vector Bundles

Thom Space of a Vector Bundle

(Assume X is paracompact)

Construction #1: $V \to X$, let $S_*(V)$ be the fiberwise 1 point compactification.

$s: X \to S_*(V)$ be the section at ∞.

$$Th(V) = S_*(V)/s(X).$$

Key: $Th(V \oplus \underline{\mathbb{R}}) = \Sigma Th(V)$.

Thom Spectrum: $MO(V)_n = Th(V \oplus \mathbb{R}^n)$,

$$\Sigma\, MO(V)_n \cong Th(V \oplus \mathbb{R}^n \oplus \mathbb{R}) \cong Th(V \oplus \mathbb{R}^{n+1}) = MO(V)_{n+1}$$

Construction #2: Put a metric on $V \to X$,

$$Th(V) = D(V)/S(V)$$

The Thom Isomorphism & Orientability (First Pass)

Recall: $\tilde{H}^*(X_+) \xrightarrow{\sim} \tilde{H}^{*+k}(\Sigma^k X_+) = \tilde{H}^{*+k}(Th(\underline{\mathbb{R}^k}))$.

Note that $H^n(E, E \setminus B) \cong H^n(S(E), B) = \tilde{H}^n(T(V))$

Thm: $p: V \to X$ be a rank n vector bundle. Let $X \hookrightarrow V$ via the zero section. Then

Pt 1 $\{$ $\exists$ a class $u \in H^n(V, V \setminus X, \mathbb{F}_2)$ whose restriction to each fiber is the orientation class in $H^n(F, F \setminus \{0\}, \mathbb{F}_2)$.

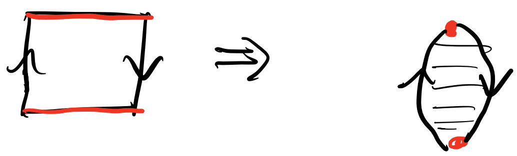
Pt 2 $\{$ Then

$$H^k(X, \mathbb{F}_2) \xrightarrow{\sim} H^{k+n}(V, V \setminus B, \mathbb{F}_2)$$
$$x \longmapsto p^*(x) \cup u$$

* If V is oriented, we can replace $\mathbb{F}_2$ by any ring! (And for $R = \mathbb{Z}$, the converse is true!)

A Quick Example

Let $M \to S^1$ be the Mobius bundle (Tautological bundle over $\mathbb{RP}^1$) we can construct $T(M)$



$= \mathbb{RP}^2$

Notice that the integral Thom isomorphism fails, but is okay in $\mathbb{F}_2$.

The Universal Thom Spectrum

Rank k $\mathbb{R}$-bundles

$$\{V \to X\}/_{\text{Iso}} \Longleftrightarrow [X, BO(k)],$$

where the isomorphism is the pullback of the universal bundle $\ell_k \to BO(k)$.

Taking $f : BO(k) \hookrightarrow BO(k+1)$,

$$f^*(\ell_{k+1}) = \ell_k \oplus \underline{\mathbb{R}}.$$

This yields a bundle map

$$\ell_k \oplus \underline{\mathbb{R}} \longrightarrow \ell_{k+1}.$$

After taking Thom spaces, we get

$$S^1 \wedge Th(\ell_k) \xrightarrow{\cong} Th(\ell_k \oplus \underline{\mathbb{R}}) \to Th(\ell_{k+1}).$$

We define $TO_k := Th(\ell_k)$ w/ the above structure maps

MO & ring structure

This is partially convention ("Spectrification")

$$MO_n = \varinjlim_k \Omega^k TO_{n+k}.$$

This forces the maps $MO_n \to \Omega MO_{n+1}$ to be equivalences without changing the (stable) homotopy groups of TO_n

The maps $BO(n) \times BO(m) \to BO(n+m)$ induce maps

$$Th(\ell_n) \wedge Th(\ell_m) \to Th(\ell_{m+n}),$$

and thus

$$MO(n) \wedge MO(m) \to MO(n+m),$$

$\Rightarrow \pi_* MO$ is a graded-commutative ring.

* $\pi_* MO = \mathbb{F}_2[u_i \mid |u_i| = i,\ i \neq 2^r - 1]$. *

We will see that it is an $\mathbb{F}_2$-algebra soon.

II: Pontryagin–Thom Construction

<u>Motivating Question (ahistorical?):</u>

What (co)-homology theory does MO represent?

Partial answer: Unoriented cobordism $\rightsquigarrow MO$.

<u>Main Tool:</u> Pontryagin–Thom Collapse Map:

Let M^n be a compact manifold.

$M \hookrightarrow \mathbb{R}^{n+k}$ extending to an embedding of a tubular neighborhood T that is the disk bundle of the normal bundle ν.

Then $\bar{T}/\partial\bar{T} \cong Th(\nu)$. Define

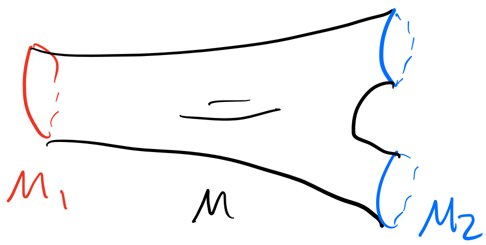
$\eta: S^{n+k+1} \longrightarrow Th(\nu)$

given by the identity on T, and sending everything else to the base point of $Th(\nu)$.

<u>Rmk:</u>

The Gauss map $M \ni x \mapsto T_xM^{\perp} \in Gr_k(\mathbb{R}^{n+k}) \subseteq BO_k$ classifies ν.

Review of Classical unoriented Cobordism

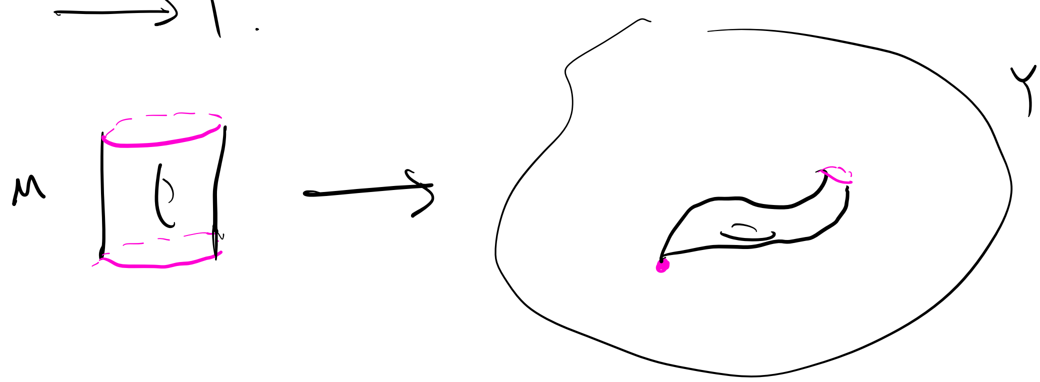


Def: two smooth n-manifolds M_1, M_2 are cobordant if there is an $(n+1)$-manifold M with $\partial M = M_1 \sqcup M_2$.

$\mathfrak{N}_n = [M^n]$ = cobordism class of n-manifolds.

$\mathfrak{N}_* = \bigoplus_{i=0}^{\infty} \mathfrak{N}_n$ is a graded $\mathbb{F}_2$-algebra under $\sqcup, \times$

$N_*(Y)$ cobordism classes of maps $M \longrightarrow Y$.

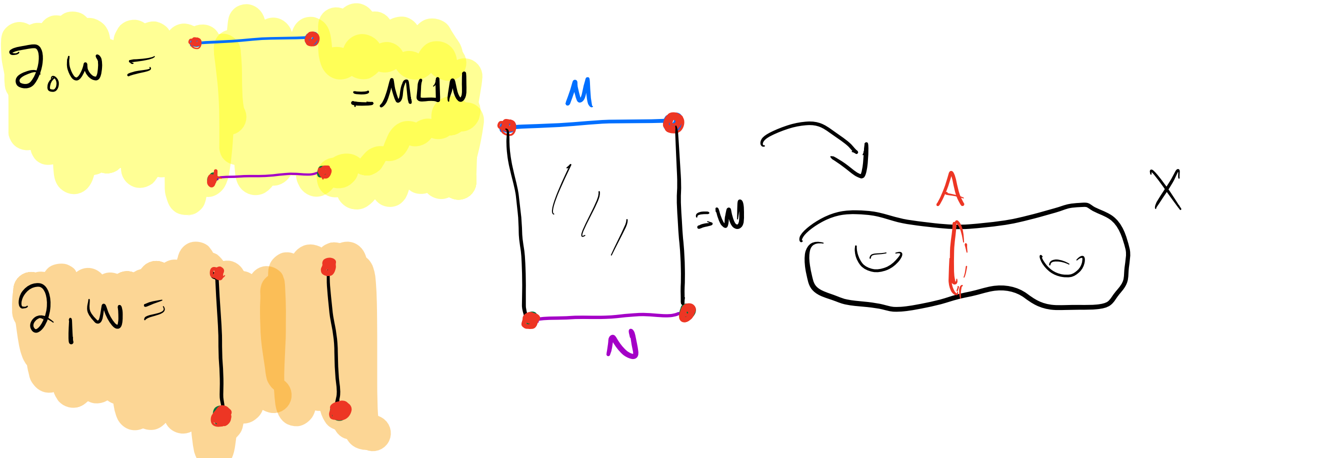


Cobordism $\Longrightarrow$ Homology $\quad N_*(X,*) = \tilde{N}_*(X)$

$N_*(X,A)$: Cobordism classes of maps

$$(N,\partial N) \longrightarrow (X,A) \sim (M,\partial M) \longrightarrow (X,A)$$

if $\exists \quad (W, \partial_0 W, \partial_1 W)$



$$\partial_0 W \cap \partial_1 W = \partial(\partial_0 W) = \partial(\partial_1 W)$$

$$\partial_0 W \cup \partial_1 W = \partial W.$$

<u>Note:</u> $N_k(X,A) \xrightarrow{\partial} W_{k-1}(A)$

is well-defined

$$\cdots \to N_k(A) \longrightarrow N_k(X) \longrightarrow N_k(X,A) \longrightarrow N_{k-1}(A) \to \cdots$$

$\rightsquigarrow$ We have a Homology Theory.

The Representing Spectrum

Claim: $\mathcal{N}_* \cong \pi_*(MO)$ [MO represents unoriented Bordism]

$M^k \hookrightarrow \mathbb{R}^{q+k}$ w/ normal bundle ν & Thom-Space $T\nu$

Pontryagin-Thom:

$$S^{q+k+1} \longrightarrow \underbrace{T\nu \longrightarrow TO(q)}_{\text{covers the classifying map.}}$$

Gives map $\mathcal{N}_k \longrightarrow \pi_k MO$.

Sketch of Well-definedness:

$\hookrightarrow \mathbb{R}^{n+k} \times [0,1]$ for k-large.

$$\begin{tikzcd}
M \arrow[r, hook, "\nu_1"] \arrow[d, hook] & \mathbb{R}^{n+k} \times \{0\} \arrow[d, hook] \\
W \arrow[r, hook, "\nu"] & \mathbb{R}^{n+k} \times [0,1] \\
M' \arrow[u, hook] \arrow[r, hook, "\nu_2"] & \mathbb{R}^{n+k} \times [0,1] \arrow[u, hook]
\end{tikzcd}$$

Do Pontryagin-Thom fiberwise to get a homotopy.

Inverse of Pontryagin Thom (Sketch)

$Gr(n,k)$

Conversely: $[f: S^{q+k} \to TO(q)] \in \pi_k MO$

make f transverse to the zero section $\quad \mathrm{codim}\, k \quad \overset{\cup}{k}$

of $TO(q) \to BO(q)$.

$\Rightarrow \quad M = f^{-1}(BO(q))$ is a k-submanifold

of S^{q+k}

($N(M)$ is pullback of classifying map!)

$g \simeq f : S^{q+k} \to TO(q)$

via $\quad F: S^{q+k} \times I \to TO(q)$

$F|_{S^{q+k}\times\{0\}}^{-1}(BO(q))$

$N \simeq f^{-1}(BO(q))$

$M \simeq g^{-1}(BO(q))$ $F|$

$W \simeq F^{-1}(BO(q))$

$\partial W = N \sqcup M$

$\partial W = F|_{S^{q+k}\times\{0\}\cup\{1\}}^{-1}(BO(q))$

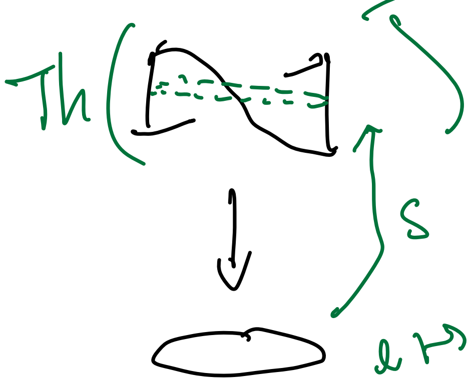
W is a cobordism, provided that we

can make F transverse.

$\Rightarrow \quad \pi_*(MO) \cong N_*$ (as rings!)

$[x:y] \in \mathbb{RP}^1$

$$\begin{aligned}
Gr_{1,2} &\cong \mathbb{RP}^1 = S^1 \\
&\vdots \quad \{(\ell, x) \in \mathbb{RP}^1 \times \mathbb{R}^2 \mid x \in \ell\} \xrightarrow{\pi} \ell \in \mathbb{RP}^1 \\
&= \{[x:y], (z,w) \in \mathbb{RP}^1 \times \mathbb{R}^2 \mid [z:w] = [x:y]\} \\
&= \{\theta, (x,y) \in S^1 \times \mathbb{R}^2 \mid \tan(y/x) = \theta\}
\end{aligned}$$



$= \mathbb{RP}^2$ all lines in $\mathbb{R}^2$

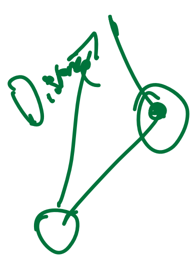
$ax + by = c$

$[0:0:1] \in \mathbb{RP}^2$

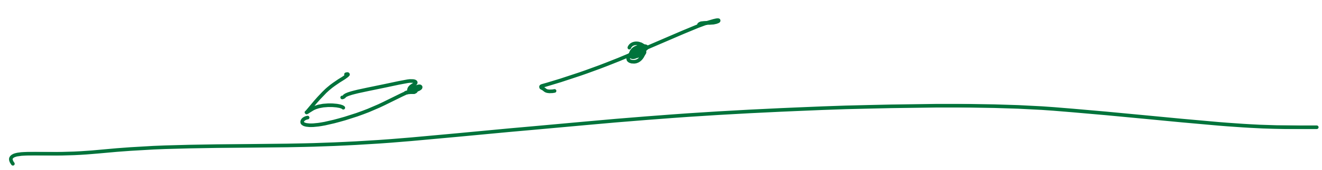
$\mathbb{RP}^2 \setminus [0:0:1]$.

s

$\ell \mapsto (\ell, \vec{0})$



$D: S^1 \times S^1 \longrightarrow \mathbb{R}$



$f: S^1 \longrightarrow \mathbb{RP}^2$

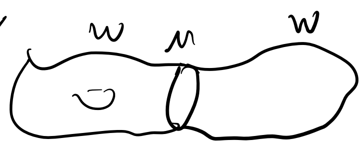
$f^{-1}\{[x:y:0]\}$

A short fun example

We can find nontrivial elements of $\pi_* MO$.

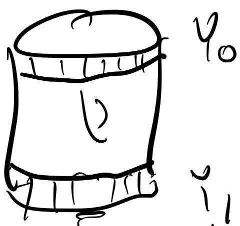
For example $[\mathbb{RP}^{2n}]$ are all nontrivial generators. Note that their Stiefel-Whitney #'s are nonzero, or: $\chi(\mathbb{RP}^{2n}) = 1 \neq 0$.

i.e: If M bounds W, $\chi(M)$ is even.

$0 = \chi\Big($ 

$\Big) = \chi(W) + \chi(W) - \chi(M)$.

compact in odd-dim.



Framed Bordism & $\pi_*(\mathbb{S})$

Framed Cobordism: $Y_0, Y_1 \subseteq M$ framed.

1) X a framed cobordism if X is itself framed w/ boundary of $[0,1] \times M$ s.t. $X \cap (\{i\} \times M) = Y_i$.

2) collar neighborhoods of boundary preserve framing

Pontryagin-Thom: $\underset{\text{(cohomotopy)}}{[M, S^n]} \xrightarrow{\sim} \Omega^{fr}_{m-n, M}$

$$\Rightarrow \quad \pi_{n+k}(S^n) = [S^{n+k}, S^n] = \Omega^{fr}_k .$$

Complex Cobordism $\rightsquigarrow$ MU

Oriented Cobordism $\rightsquigarrow$ MSO

Spin Cobordism $\rightsquigarrow$ $MSpin$

$\vdots$

orientations on Vector Bundles

- For V a vector space, an orientation on V is a generator of $H^n(V, V\setminus\{0\}, \mathbb{Z}) = \mathbb{Z}$.
(orientation preserving maps induce the identity).

- An <u>orientable</u> vector bundle $Y \to X$ is one w/ a class $u \in H^n(Y, Y\setminus Y_0, \mathbb{Z})$ whose restriction to each fiber is the orientation class in $H^n(V, V\setminus\{0\}, \mathbb{Z})$

$$H^n(Y, Y_0, \mathbb{Z}) = \tilde{H}^n(Th(V), \mathbb{Z}).$$

E- orientations on Vector Bundles

Def: Let E^* be a mult. cohomology theory. A vector bundle $V \to X$ is E-orientable if we have

$$u \in \tilde{E}^n(Th(V))$$

such that for all $x \in X$ & inclusions

$$i_x^* : \tilde{E}^n Th(V) \to \tilde{E}^n(S^n),$$

$i_x^*(u)$ is a unit in $\tilde{E}^n(S^n) \cong \tilde{E}^0(S^0)$.

Equivalently: For E a ring Spectrum, a map $u : Th(V) \to E$ such that the map of E-modules

$$E \wedge Th(V) \to E \wedge Th(V) \wedge X_+ \xrightarrow{E \wedge u \wedge X_+} E \wedge E \wedge X_+ \longrightarrow E \wedge X_+$$

is an isomorphism. $\cong$

$E = H\mathbb{Z}$

(so, $\tilde{E}^*(X_+) \cong \tilde{E}^* Th(V)$)

Codify the Thom isomorphism.

Complex Orientations

Def: A cohomology Theory is complex-oriented if there is a choice of Thom class u_V for all complex vector bundle $E \to X$ that is

1) functorial under pullbacks
2) sends direct sums to products ($u_{V \oplus W} = u_V u_W$).

ex: $H\mathbb{Z}$, MU, ...

If E is complex-oriented, we get a map $MU \to E$, i.e: $u \in E^*(MU)$ & Moreover

$[\text{ring maps } MU \to E] \Leftrightarrow [\text{complex orientations on } E]$.

Remark: The situation is similar for MO, $MSpin$, MSO, ...