Review of Tate Construction

Point-Set model

Summary: $EG_+ \to S^0 \to \tilde{EG}$

$\varepsilon : F(S^0, X) \to F(EG_+, X)$

$$\begin{tikzcd}
X \wedge EG_+ \arrow[r] \arrow[d, "\varepsilon\wedge 1"', "\wr"] & X \arrow[r] \arrow[d, "\varepsilon"] & F(EG_+, X) \wedge \tilde{EG} \arrow[d, "\varepsilon\wedge 1"] \\
F(EG_+, X) \wedge EG_+ \arrow[r] & F(EG_+, X) \arrow[r] & F(EG_+, X) \wedge \tilde{EG}
\end{tikzcd}$$

G-fixed :

$$\begin{tikzcd}
X_{hG} \arrow[d, "\wr"] & & \\
X_{hG} \arrow[r, "\mathrm{norm}"] & X^{hG} \arrow[r] & X^{tG}
\end{tikzcd}$$

Observation: Thomas Tower! :)

Taking e-fixed points gives

$$\begin{tikzcd}
X \arrow[d, "?"] & \\
X \arrow[r] & X
\end{tikzcd}$$

, but we always get morphisms from G-fixed to e-fixed

$$\begin{tikzcd}
X_{hG} \arrow[r, "N"] \arrow[d, "t"'] & X^{hG} \arrow[d, "r"] \\
X \arrow[r, equal] & X
\end{tikzcd}$$

.

When $G \circlearrowright X$ trivially,

$$\begin{tikzcd}
X \wedge BG \arrow[r, "N"] \arrow[dr, "t"'] & \mathrm{Maps}(BG, X) \arrow[d, "r"] \\
& X \arrow[u, dashed, bend right, "\varepsilon"']
\end{tikzcd}$$

evaluation at a point

Outline

(Mathew-Clausen)

General Theorem: L is any Bousfield localization functor of spectra such that $\exists$ a functor

$$\Phi : S_* \to L\mathrm{Sp} \text{ with } \Phi\,\Omega^\infty \simeq L,$$

then: $L X^{tG} \simeq *$ $\forall$ G-objects $X \in L\mathrm{Sp}$.

Steps: 1) $L X^{tG} \simeq *$

$\Updownarrow$

The transfer $\Sigma^\infty_+ BG \longrightarrow \Sigma^\infty_+ *$ admits a section after applying L

2) Reduce to the case where $G = C_p$ (Kuhn)

3) (Kahn-Priddy '78): The transfer $\Sigma^\infty_+ BC_p \to \Sigma^\infty_+ *$ admits a section after applying $\Omega^{\infty+1}$.

4) If $\exists\ \Phi\ S_* \to L\mathrm{Sp}$ $\ni$

$$\Phi\, \Sigma^{\infty} \simeq L$$

$\Rightarrow$ Result

Setting $L = L_{T(n)}$, we have Φ, which is the Bousfield-Kuhn functor.

($L_{T(n)}X$ doesn't depend on a spectrum, just the underlying space.)

[GM] 1.3 R a comm ring spectrum w/ trivial G-action, M an R-module

$\Rightarrow$ R^{tG} is a ring spectrum & M^{tG} is an R-module.

Diagonal map

$\exists !$. up to homotopy equiv
$E G_+ \wedge E G_+ \simeq E G_+$, $\widetilde{EG} \wedge \widetilde{EG} \simeq \widetilde{EG}$.

$$\begin{array}{ccccc} f(X) \wedge f(Y) & \to & c(X) \wedge c(Y) & \to & t(X) \wedge t(Y) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ f(X \wedge Y) & \to & c(X \wedge Y) & \to & t(X \wedge Y) \end{array}$$

TFAE:

1) $X^{tG} = 0 \quad \forall X \in \mathrm{Fun}(G, LSp)$

$\Updownarrow$

2) $LS^{tG} = 0$ w/ trivial G-action.

LS^{tG} E_*-Acyclic $\Rightarrow$ X^{tG} is as well.

$\Rightarrow$ It will suffice to show that

$$LS^{tG} \simeq *$$

<u>Goal</u>: Relate LS^{tG} to the transfer

$$\Sigma^\infty_+ BG \longrightarrow \Sigma^\infty_+ *$$

given $EG \to BG$ G acts trivially (GM, 7.5)

$$\begin{tikzcd}
LS \wedge BG_+ \arrow[r, "N", "\sim"'] \arrow[dr, "a"] & LS^{hG} \arrow[r] \arrow[d, "r"] & LS^{tG} \\
& LS \arrow[ul, bend left, "L\varphi"] &
\end{tikzcd}$$

r: is a "choice of basepoint" in BG_+.

$$X^{hG} = \mathrm{Maps}(BG_+, LS) \longrightarrow LS.$$

r for retract, section given by
$c : BG_+ \longrightarrow S^0$,

$$\mathrm{Maps}(S^0, LS) \xrightarrow{c^*} \mathrm{Maps}(BG_+, LS)$$

transfer "sums" over the fiber in

$X \wedge S \longrightarrow X \wedge BG_+$, $\Rightarrow$ factors through X^{hG}.

If N is an equivalence

$\Rightarrow$ φ has a section:

For the converse, we need a lemma.

<u>Lemma:</u> Let R be a mult. cohomology theory, K a <u>pointed</u> connected CW-complex wrt. basepoint κ.
If $r \in R^0(K)$ restricts to a unit in $R^0(\{\kappa\})$ then r was already a unit.

<u>Pf:</u> Show $\forall$ pointed subcomplexes $K' \subset K$.

<u>Step 1:</u> Assume K' is finite

when $K' = \kappa$ ✓

otherwise AHSS

$$\begin{array}{ccc} H^q(K', R^{-q}(*)) & \Rightarrow & R^0(K') \\ \downarrow & & \downarrow \\ H^q(\{\kappa\}, R^{-q}(*)) & \Rightarrow & R^0(*) \end{array}$$

$$\ker(i^*) \overset{E_2}{\cong} \bigoplus_{q=0}^{\dim K'} H^q(K', R^{-q}(*))$$

deg 0 $\Rightarrow i^*$ is iso

deg ≥ 1, mult. raises degree,

So we get zero eventually

$* \longrightarrow K$

$\underline{R^0(K)} \to \underline{R^0(*)}$

$\Rightarrow$ $\ker(i^*)$ is nilpotent.

$\Rightarrow$ result (nilradical $\subseteq$ Jacobson radical)

$\Rightarrow$ For infinite complexes, milnor SES + five lemma.

<u>Thm:</u> $L S^{tG} \simeq *$

$\Updownarrow$

$\Sigma^\infty BG \to \Sigma^\infty *$ splits after applying L

$$\begin{tikzcd}
LS_{hG} \arrow[r, "N", "\sim"'] \arrow[dr, "a"] & LS^{hG} \arrow[r] \arrow[d, "r"] & LS^{tG} \\
& LS \arrow[ul, bend left, "L\psi"] &
\end{tikzcd}$$

Pf: only need to show:

if $L\psi$ has a section, then N is an equivalence.

It will suffice to show that

$$\pi_0(LS_{hG}) \twoheadrightarrow \pi_0(LS^{hG}),$$

i.e: the image of $\pi_0 LS_{hG} \xrightarrow{\sim} \pi_0 LS^{hG}$ contains a unit.

$L\psi$ has a section $\Rightarrow \exists\, x \in \pi_0(LS_{hG})$ whose image under $L\psi = r \circ N$ is 1.

$\Rightarrow N x \in \pi_0 LS^{hG} = LS^0(BG)$ is a unit because

$$r_N X \in \pi_0 LS = LS^0(*) = \Delta.$$

$\boxtimes$

Step 2: Reduce to the case where $G = C_p$. (Kuhn '04)

This follows from two Lemmas:

1) $G/H \longrightarrow E_*(X^{tH})$ defines a Mackey functor.

May-McClure ('82) show that if

$$E_*(X^{tH}) = 0 \ \forall \ H \trianglelefteq G \text{ of prime}$$

power order, then $E_*(X^{tG}) = 0$.

2) $K \trianglelefteq G$ normal, $Q = G/K$, R a ring spectrum, E a homology theory

$t_K(R)$, $t_Q(R)$ E_*-acyclic

$\Rightarrow t_G R$ is as well.

pf:

Klein's characterization of the norm.

Let $N_G, N_G' : Y_{hG} \to Y^{hG}$ be natural transformations $\ni$

$$N_G(\Sigma^\infty G_+) \ \& \ N_G'(\Sigma^\infty G_+)$$

are w.e.

then $\exists!$ w.e $f(Y) : Y_{hG} \to Y_{hG}$ $\ni$

$$\begin{tikzcd} Y_{hG} \arrow[r, "N_G(Y)"] \arrow[dr, "f(Y)"'] & Y^{hG} \\ & Y_{hG} \arrow[u, "N_G'(Y)"'] \end{tikzcd}$$
$\circlearrowleft$

$$\begin{tikzcd} Y_{hG} \simeq (Y_{hK})_{hQ} \arrow[r, "N_K(Y)_{hQ}"] & (Y^{hK})_{hQ} \arrow[r, "N_Q(Y^{hK})"] & (Y^{hK})^{hQ} \simeq Y^{hG} \end{tikzcd}$$

$? \ N_G(Y)$

check on $Y = \Sigma^\infty G_+$.

Since $\Sigma^\infty G_+ \simeq \bigvee_{gK \in Q} \Sigma^\infty K_+$,

$N_K(\Sigma^\infty G_+) \Rightarrow N_K(\Sigma^\infty G_+)_{hQ}$ is an equiv.

$$(\Sigma^\infty G_+)^{hK} \xleftarrow[\sim]{N_K(\Sigma^\infty G_+)} (\Sigma^\infty G_+)_{hK} \xrightarrow[\sim]{} \Sigma^\infty Q_+$$

$\Rightarrow N_Q((\Sigma^\infty G_+)^{hK})$ is an equivalence.

$\Rightarrow$ $N' = N_Q(R^{hK}) \circ N_K(R)_{hQ}$,

$\mathrm{cofib}(N') \simeq R^{tG}$

$N_K(A)$ is an E_* isomorphism

$\Rightarrow N_K(A)_{hQ}$ is too (not vacuous, but true)

$\Rightarrow$ R^{tQ} is E_*-acyclic,

$(R^{h\mathbb{F}})^{tQ}$ is an R^{tQ}-module,

so it is also E_*-acyclic

$\Rightarrow$ Lemma.

<u>Conclusion</u>:

$$LX^{tG} \simeq 0 \quad \forall\ X \in \mathrm{Fun}(BG, Sp).$$

$$\Uparrow$$

$\Sigma^\infty_+ BC_p \longrightarrow \Sigma^\infty_+ *$ admits a

section $\forall\ p$ after applying L.

Thm Kahn-Priddy: $QBC_p \to \Sigma^\infty_+ *$ admits a section.

(Mathew-Clausen)

<u>Main thm</u>: If $\exists$ $\Phi : S_* \to L\,Sp$ $\ni$ $\Phi \circ \Omega^\infty(X) \simeq LX$, then

$L X^{tG} \simeq *.$

Now, let $L = L_{T(n)}$. We want to explain how we can obtain such a Φ, (Bousfield–Kuhn functor.)

<u>Step 1:</u> Define $\Phi_v : S_* \to Sp$.

Let B be a finite space of type n w/ a v_n self-map

$$\Sigma^d B \to B.$$

$k \equiv e \mod d$

$$\Phi_v(X)_k = \Omega^e \operatorname{Map}_{S_*}(B, Z)$$

if $k \equiv 0 \mod d$,

$$v(Z) : \operatorname{Map}(B, Z) \xrightarrow{v^*} \operatorname{Map}(\Sigma^d B, Z) = \Omega^d \operatorname{Map}(B, Z)$$

Fact #1: $\Phi_v(X)$ is $T(n)$-local

$$[F, \Phi_v(Z)] = 0 \quad \forall\ F \in C_{n+1}.$$

$\Phi_v(X)$ periodic, assume $F = \Sigma^\infty A$ for some space A of type $\geq n+1$.

$$[F, \Phi_v(Z)] = \pi_0(\mathrm{Maps}(A, \Phi_v(Z))) = 0$$

since

$$\mathrm{Maps}(A, \Phi_v(Z)) = \Phi_{1_A \wedge v}(Z) \simeq *$$

since $A \wedge B$ will have typ ≥ 1, so

$1_A \wedge v : \Sigma^d A \wedge B \to A \wedge B$ is nilpotent.

Fact #2: In general, for any map $\Sigma^d B \to B$

$$\Phi_v(\Omega^\infty E) \simeq v^{-1}\,\mathrm{Maps}(B, X)$$

$$= \mathrm{colim}\left(\mathrm{Maps}(A, X) \xrightarrow{u^*} \mathrm{Maps}(\Sigma^d A, X) \to \cdots\right)$$

when B is finite of type n,

$$\Phi_v(\Omega^\infty E) \simeq \mathrm{Maps}(B, L_{T(n)}(X))$$

<u>Fact #3</u>: Does not depend on the choice of map

<u>Def</u>: Let C_t be the ∞-category whose objects are pointed spaces, w/ a v_n self map $v: \Sigma^t V \to V$

type n

<u>Key observation</u>: $s, t > 0$ $\exists$ a functor

$$C_t \longrightarrow C_{st}$$

$$(V, v) \longmapsto (V, v^s)$$

$$v^s = \Sigma^{st} V \xrightarrow{\Sigma^{(s-1)t} v} \Sigma^{(s-1)t} V \to \cdots \xrightarrow{v} V$$

$$\begin{tikzcd} C_t \arrow[r] \arrow[dr] & C_{st} \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Fun}(S_*, Sp) \end{tikzcd}$$

&

$$\begin{tikzcd} \mathcal{C}_t \arrow[rr, "\varepsilon"] \arrow[dr, "\Phi_\bullet"'] & & \mathcal{C}_t \arrow[dl, "\varepsilon\Phi_\bullet"] \\ & \mathrm{Fun}(S_*, Sp) & \end{tikzcd}$$

$\Rightarrow$ we can combine all of these into

C', where

$$C' = \lim \quad C_{1!} \to C_{2!} \to C_{3!} \to \cdots$$

$$C_{(m-1)!} \to C_{m!} \quad \text{is}$$

$$(V, v) \mapsto (\varepsilon V, \varepsilon(v^m))$$

obtain $\Phi : C' \to \mathrm{Fun}(S_*, Sp)$

$$C' \simeq Sp^{fin}_{\geq n}$$

$\Rightarrow$ we have

$$\Phi_\bullet : (Sp^{fin}_{\geq n})^{op} \longrightarrow Fun(S_*, Sp)$$

(idea: E a finite spectrum of type n, choose k, where $\Sigma^k E \simeq \Sigma^\infty V$, V is a finite space type n w/ v_n self-map.)

$$\Phi_E = \Sigma^k \circ \Phi_V$$

Kan extend!

$F : (Sp^{fin})^{op} \longrightarrow Fun(S_*, Sp)$ is

the right Kan extension of $E \to \Phi_E$,

$$\Phi : S_* \longrightarrow Sp \; := F(S),$$

$$\Phi(X) = \varprojlim_{E \to S} \Phi_E(X)$$

$$\Phi(\Omega^\infty X) = \lim_{E} \Phi_E(\Omega^\infty X)$$

$$= \lim_{E} \mathrm{Maps}(E, L_{T(n)} X) = Y$$

$L_{T(n)} X \wedge E^\vee$ $\quad E \to S$

$$\approx L_{T(n)} X$$

$S \to E^\vee$

$S \to \cdots \to S/p^3 \to S/p^2 \to S/p$

$\mathrm{Maps}(S, L_{T(n)} X) \to \mathrm{Maps}(E, L_{T(n)}$

$$Y = \lim_{\substack{\leftarrow \\ n}} (L_{T(n)} X)/p^n$$

$$L_{T(1)} X = (L_1^f(X))^\wedge_p$$

$$S^k/p \xrightarrow{f} S/p$$

$$\mathrm{colim}\left(X/p \xrightarrow{f} \Sigma^{-k} X/p \xrightarrow{f} \Sigma^{-2k} X/p \to \cdots\right)$$

$$|| $$

$$(X/p)[v_1^{-1}]$$

$$\lim_{\leftarrow}\left(\cdots \to (X/p^3)[v_1^{-1}] \to (X/p^2)[v_1^{-1}] \to (X/p)[v_1^{-1}]\right)$$

$$||$$

$$L_{T(1)} X$$

$T(n)$-vanishing for X^{tG}. Optional

Main result: $L_{T(n)} X^{tG} \simeq *$
for all $X \in \mathrm{Fun}(BG, Sp)$

Motivation:

Originally: Kuhn ('04) proved this to show that Goodwillie tower splits $K(n) \otimes T(n)$ locally.

$F: Sp \to Sp$

$$\begin{tikzcd} & P_n F(X) \arrow[d] \\ & P_{n-1} F(X) \arrow[d] \\ F(X) \arrow[uur] \arrow[ur] & \vdots \end{tikzcd}$$

$D_d F(X) := \mathrm{fib}(P_d(X) \to P_{d-1} X)$

$D_d F(X) \simeq (\Delta_d F X)_{h\Sigma_d}$

F commutes w/ directed homotopy colimits

$D_d F(X) \simeq (C_d \wedge X^{\wedge d})_{h\Sigma_d}$

$\exists$ pullback

$$\begin{tikzcd} P_d F(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Delta_d F(X)^{h\Sigma_d} \arrow[d] \\ P_{d-1} F(X) \arrow[r] & t_{\Sigma_d}(\Delta_d F(X)) \end{tikzcd}$$

$\rightsquigarrow$ Can be used to show

$$L_{T(n)} P_d F(X) \simeq \prod_{c=0}^{d} L_{T(n)} D_c F(X)$$

is $(ex_!^*)$ invertible

$\Updownarrow$

n-ambidexterous

$$D_n(x) = \left(C_n \wedge x^{\wedge n}\right)_{\Sigma_n}$$

$$\frac{c_n x^n}{n!}$$

Tobias

"Z-segal spaces"