# Background & Motivation.

- Geometry: Clifford - Klein space form is a closed Riemannian manifold of constant sectional curvature

- Classical:
  quotients of $S^n$, $\mathbb{R}^n$, $\mathbb{H}^n$ by discrete groups acting by isometries. (free + properly discontinuously)

- Spherical space forms completely classified. Reduced to algebra:

finite $G$ w/ rep $\varphi : G \to O(n+1)$

$\varphi(g)$ has no $+1$ eigenvalue for all $g \in G$. (this would be a fixed point.)

ex: $\mathbb{Z}/n \curvearrowright S^1$ $\forall$ $n$ freely + orthogonally

$Q(4k) = \{x, y \mid x^{2k} = 1, x^k = y^2, yxy^{-1} = x^{-1}\}$

act on $S^3$ by quaternionic multiplication.

1925: Hopf asked about topological space forms, i.e.:

Manifolds having $S^n$ as its universal cover

(what groups $G$ act freely on $S^n$ for some $n$?)

* (This problem can be sharpened by finding such $n$ in terms of $G$)

We will focus on Swan's result: (1960)

Thm: A finite group $G$ acts freely on a finite complex

$X \simeq S^n \iff$ every Abelian subgroup of $G$ is cyclic

This can be sharpened by surgery theory:

(Madsen - Thomas - Wall) $G$ acts freely on a sphere $\iff$

(1) All abelian subgroups are cyclic
no subgroups of the form ($\mathbb{Z}/p \times \mathbb{Z}/p$) "$p^2$ condition."

(2) Every subgroup of order $2p$ is cyclic, "$2p$ condition"
// $\Rightarrow D_{2n}$ cannot act freely on spheres.

$\Rightarrow D_{2n}$ can act on finite complexes $X \simeq S^k$ freely!

Swan, Cartan-Eilenberg, Milnor

We will show the necessity of (1), (2) & the full theorem of Swan in this talk:

$\mathbb{Z}/2 \circlearrowright S^n$ freely $\forall$ $n$ by antipodal action

## First Obstructions:

Lefschetz-fixed point thm:

$X$ a triangulable space, then

$f: X \to X$,

$$\Lambda_f := \sum_{i \geq 0} (-1)^i \, tr\left(f_* \mid H_i(X, \mathbb{Q})\right).$$

If $\Lambda_f \neq 0 \Rightarrow f$ has a fixed point.

Any fixed point free map $g: S^k \to S^k$ homotopic to antipodal map

$$\Rightarrow \quad deg(g) = (-1)^{k+1}$$

$$\Lambda_g = 1 + (-1)^k (-1)^{k+1}$$

$$= 1 + (-1)^{2k+1}$$

Thus for any $g, h \in G$

$$\Lambda_{gh} = 1 + (-1)^k (-1)^{2k+2}$$

$$= 1 + (-1)^k.$$

Thus, if $k$ is even,

$\Lambda_{gh} \neq 0$, and so

$G = \mathbb{Z}/2$

If $|G| > 2$, $G \curvearrowright S^k$ freely, then $k$ is odd.

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Let's see the "2p" condition (every subgroup of order $2p$, $p > 2$ is cyclic.)

<u>Thm (Milnor)</u>: Let $T : S^n \to S^n$ be an involution w/o fixed points. $f : S^n \to S^n$ a map of odd degree. Then $\exists$ $x \in S^n$ $Tf(x) = fT(x)$.

Proof of "2p": i) Let $H \triangleleft G$, $|H| = 2p$, $H$ not cyclic

$\Rightarrow H$ is Dihedral. Let $T$ be the action of order 2 generator.

$f$ is action of order $n$ generator.

$\Rightarrow T f T^{-1} f^{-1}$ has a fixed point,

so $T f T^{-1} f^{-1} = Id$ $\Rightarrow$ generators commute

$\Downarrow$

Proof of necessity "$p^2$" condition

(All Abelian subgroups are cyclic.)

A free resolution of period $n$:

$$0 \to \mathbb{Z} \xrightarrow{\mu} F_{n-1} \to \cdots \to F_1 \to F_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

of $\mathbb{Z}[G]$ modules, $F_i$ free, $G \curvearrowright \mathbb{Z}$ trivial d

**Proposition**: $G \curvearrowright S^{n-1}$ freely, $n-1$ odd, $G$ has a free resolution of period $n$.

- $S^{n-1}/G \simeq$ CW complex $X$, $\dim X = n-1$

- $\tilde{X} \simeq S^{n-1}$

- $C_i(\tilde{X})$ are free $\mathbb{Z}[G]$-modules.

All maps $S^{n-1} \to S^{n-1}$ have degree $1$, so $G \curvearrowright H_*(\tilde{X}) = H_*(S^{n-1})$ is trivial.

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- $0 \to \mathbb{Z} \to C_{n-1}(\tilde{X}) \to \cdots \to C_0(\tilde{X}) \to \mathbb{Z} \to 0$

$C_{n+1}$ $C_n$

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$0 \xrightarrow{\partial_n} C_{n-1} \to C_{n-2}$

**Recall**:

$|G| < \infty$ ; $M$ a $\mathbb{Z}[G]$-module.

$$H^n G(G,M) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

free $\mathbb{Z}[G]$-resolution

$$\to F_n \to F_{n-1} \cdots \to F_0 \to \mathbb{Z} \to 0$$

$$\cdots \leftarrow \mathrm{Hom}(F_1, M) \leftarrow \mathrm{Hom}(F_0, M) \leftarrow$$

$$H^n(G, M)$$

$$0 \leftarrow \mathbb{Z} \leftarrow \mathrm{Hom}(C_{n-1}(X), \mathbb{Z})$$

<u>Corollary</u>: $G \curvearrowright S^{n-1}$ w/ $n-1$ odd,

a) $\hat{H}^n(G, \mathbb{Z})$ is periodic w/ period $n$.

b) all abelian subgroups are cyclic
(no abelian subgroups $\mathbb{Z}/p \times \mathbb{Z}/p$)

Pf: c) $\to F_0 \xrightarrow{\mu\varepsilon} F_{n-1} \to \dots \to F_0 \xrightarrow{q\varepsilon} \dots$

$\Rightarrow$ periodic cohomology

b) Follows from a Kunneth formula for group cohomology applied to $H^*(\mathbb{Z}/p \times \mathbb{Z}/p, \mathbb{Z})$
& (a)

In fact $b \Leftrightarrow a$ as well. From now on, we will consider the condition that $H^n(G, \mathbb{Z})$ is periodic.

(remark: This can be done w/ the Serre spectral sequence: Cartan & Eilenberg on $S^n \to X \to K(G,1)$.)

Swan showed in 1960 that if $G$ satisfies the $p^2$ condition then we can find a finite periodic resolution by projective $\mathbb{Z}[G]$-modules

$$0 \to \mathbb{Z} \to P_n \to \cdots \to P_0 \to \mathbb{Z} \to 0.$$

We already saw that free is necessary for $G \curvearrowright \tilde{X}$ freely w/ $\tilde{X} \simeq S^n$ & $\tilde{X}$ a finite CW complex.

Algebraic K-theory

Let $R = \mathbb{Z}\pi_1 X$.

Let $P_R = \{$iso classes of f.g. projective $R$-modules$\}$

$P_R$ is a monoid under $\oplus$.

$K_0(R) = gr(P_R)$

i.e: we force $[P \oplus P'] = [P] + [P']$.

$K_0(R) \longrightarrow K_0(S)$ induced by $(- \otimes_R S)$

$\tilde{K}_0 = K_0(S)/K_0(\mathbb{Z})$ ;

Mod out by free things:

$[R^n] = [0]$. Eilenberg Swindle

why f.g.? $\forall$ proj. $P, \exists M \ni P \oplus M \cong R^n$.

$F = M \oplus P \oplus M \oplus P \oplus \cdots$

$P \oplus F = P \oplus M \oplus P \oplus \cdots \cong F$ is free

<u>Euler characteristic map:</u>

$\chi : \{\text{f.g. projective complexes}\} \longrightarrow K_0 R$

$$\chi(P_n \to \cdots \to P_0) = \sum_{i \geq 1}^{n} (-1)^i [P_i]$$

<u>Two Key facts:</u>

1) $P_n \to \cdots \to P_0$ is chain homotopic to $F_n \to \cdots \to F_0$.

$\Rightarrow \chi(P_\bullet) = 0.$

Let $G_0$ be a module s.t.

$P_0 + G_0 = F_0$ is free. Then

$$\begin{array}{l} P_n \to \cdots P_1 \to P_0 \\ 0 \cdots \to 0 \to Q_0 \xrightarrow{\sim} Q_0 \quad \oplus \\ \hline P_n \to \cdots \to P_1 \oplus Q_0 \to F_0 \end{array}$$

Continue by induction. To $(n-1)$.

Since $\chi(P_\bullet) = 0$, if

$P_{n-1}, \ldots, P_0$ are free,

$P_n$ must be stably free,

since $\chi(P_\bullet) = (-1)^n [P^n] + 0.$

Fact #2: $|G|<\infty$

$\Rightarrow |\tilde{K}_0(\mathbb{Z}[G])| < \infty.$

Thus, starting with

$$\mathbb{Z} \to P_n \xrightarrow{\mu} P_{n-1} \to \cdots \to P_0 \xrightarrow{\varepsilon} \mathbb{Z},$$

$$P_{2\bullet} = P_n \to \cdots \to P_0 \xrightarrow{\mu\varepsilon} P_n \to \cdots \to P_0$$

$$\chi(P_{2\bullet}) = 2 \cdot \chi(P_\bullet) \in \tilde{K}_0(\mathbb{Z}[G])$$

$\exists\, k \in \mathbb{N}$ w/

$$\chi(P_{k\bullet}) = 0 \in \tilde{K}_0(\mathbb{Z}[G]).$$

Thus we can find a periodic resolution of $G$ w/

$\mathbb{Z} \to F_m \to \cdots\cdots \to F_1 \to F_0 \to \mathbb{Z}$,

$m >>> 0$, $F_i$ f.g. free $\mathbb{Z}[G]$-modules

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let $X^{(2)}$ be a finite CW complex w/ $\pi_1(X^{(2)}) = G$.

let $C_i = C_i(\tilde{X})$.

By a lemma of Schanuel:

$D = F_2 \oplus C_1 \oplus F_0$

$E = C_2 \oplus F_1 \oplus C_0$

$\ker(C_2 \to C_1) \oplus D$

$\cong \ker(F_2 \to F_1) \oplus E$

$\Downarrow$

we can splice

$0 \to \mathbb{Z} \to F_{n-1} \to \cdots \to F_4$

$\to F_3 \oplus E \to C_2 \oplus D \to C_1 \to C_0 \to \mathbb{Z} \to 0$

$\forall$ generators of $D$, add a 2-cell to $X^{(2)}$ to get $Y^{(2)}$.

$\forall$ $\alpha$ of $F_3 \oplus E$

add a 3-cell to $Y^{(2)}$ via

$$\partial(\alpha) \in H_2(\tilde{Y}^{(2)}) = \pi_2(\tilde{Y}^{(2)})$$
$$= \pi_2(Y^{(2)})$$

continue inductively for the results,

$\tilde{Y}^{(m)} \simeq S^m$, $\pi_1(Y^{(m)}) = G$.

# Remark on Eilenberg-Sconvelle !!!

PUNCHLINE:

$\nexists$ obstruction for $G$ satisfying $p^2$ condition to act freely on $X \simeq S^n$ for $n \geq 3$.

Remark - calculating a tight $n$ has been done (hard.)

Topology $\Longleftrightarrow$ Algebra
(Wall) (Swan)

Swan actually did a lot more:

a $(G,n)$-polarized space is a pointed $(n-1)$-complex $X$, that is <u>finitely dominated</u>, (retract of a finite CW-complex)

$\pi_1(X, x_0) = G$ & $\tilde{X} \simeq S^{n-1}$.

$$
\begin{tikzcd}
 & P_{n-1} \arrow[d, dotted] & K(\mathbb{Z}, n-1) \arrow[l] \\
 & K(G,1) \arrow[d] & \\
X \arrow[uur] \arrow[ur] \arrow[r] & K(G,1) &
\end{tikzcd}
$$

$$K(\mathbb{Z}, n-1) \longrightarrow P_{n-1} \longrightarrow \cdots \to K(G,1)$$

$$\begin{tikzcd}
X_{n-2} \arrow[r, "f"] \arrow[d] & Y \arrow[r] & P_1(Y) \arrow[d] & \\
X_{n-1} \arrow[rr] \arrow[rrr, bend right] & & K(G,1) \arrow[r, "k_1"] & K(\mathbb{Z}, n)
\end{tikzcd}$$

$\pi_{n-1}Y$

$K(G,1) = \pi_1 Y$

$$[K(G,1), K(\mathbb{Z},n)] = H^n(K(G,1), \mathbb{Z})$$
$$= H^n(G, \mathbb{Z}).$$

Periodicity: $H^{n+1}(G, \mathbb{Z}) = \mathbb{Z}/|G|$, two gen. of $\mathbb{Z}/|G|$ differ by an element $\pm \operatorname{im}(\operatorname{Aut} G \to \mathbb{Z}/|G|)$.

g also gives a proj. resolution

$$\cup \qquad \mathbb{Z} \to P_{n-1} \cdots P_0 \to \mathbb{Z}.$$

Conversely, Swan showed that $\forall$ generators $g \in H^{n+1}(G, \mathbb{Z})$, $\exists$ a polarized $(G, n-1)$ complex $X_g$

k-invariant is $g$.

(our argument from earlier!!, applied to $P_n \oplus F_n = R^n$, $F_n$, not f.g.

The Swan obstruction

$$\chi(P_\bullet) \simeq \sigma(X_g),$$

where $\sigma(X_g) \in \tilde{K}_0(\mathbb{Z}[G])$ is the Wall-finiteness obstruction:

$\sigma(X_g) = 0 \iff \sigma$ has the homotopy type of a CW-complex.

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