Diff.

*This talk owes a lot to Sander Kupers

Part 1: Diffeomorphisms of disks

Part 2: Some high-dim mflds (weird)

Part 3: Some friends of $\mathrm{Diff}_\partial(D^n)$

Part 4: Algebraic K-Theory???

Def: $\mathrm{Diff}_\partial(D^n)$ is the group of C^∞-Diffeomorphisms of D^n that are the identity on a nbhd of ∂D^n, in the C^∞-Topology (or compact-open)

Note: $\pi_0 \mathrm{Diff}_\partial(D^n) = \mathrm{Diffeo}/\mathrm{isotopy}$

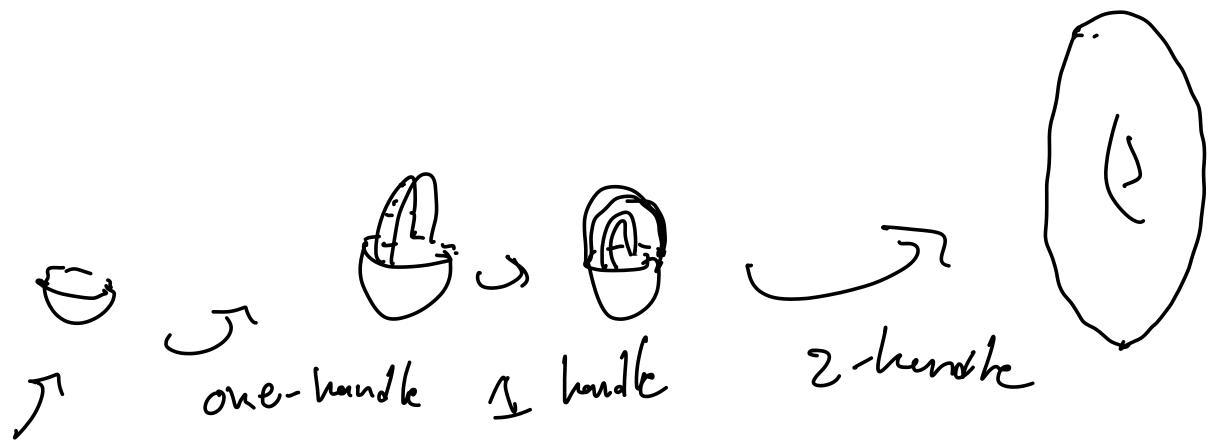
isotopy: homotopy through diffeomorphisms

Why study Mfld Topology of Disks?

In high Dim Mfld theory, we decompose Mflds into their constituent parts:

Handles:

k-handle: $D^k \times D^{n-k}$, glue by $\partial D^k \times D^{n-k}$



0-handle   one-handle   1 handle   2-handle

Disks are the building Blocks of high dim mflds

Reason #2: Diffeomorphisms of Disks

Measure subtle aspects of smoothing Theory:

Thm (Alexander Trick):

$\mathrm{Homeo}_\partial(D^n) \simeq * \quad \forall n.$

$$J(x,t) = \begin{cases} t f(x/t), & \text{if } 0 \le \|x\| < t \\ 1 & t \le \|x\| \le 1 \end{cases}$$

Bill Thurston: "combing all the tangles to one point"

straighten from the boundary.

* TOP mflds: push all issues to a point
* $\mathrm{diff}(\mathbb{R}^n)$: Push all issues to ∞.

$\mathrm{diff}_\partial D^n$ measures interesting phenomena.

Low dimensions:

d = 1: $\mathrm{Diff}_\partial(D^1) \simeq *$.

$f: [0,1] \to [0,1]$ fixing the boundary

is a diffeo

$\Leftrightarrow \frac{\partial f}{\partial t} > 0$ everywhere.

"Tightening the string"

$$H_t(f(0)) = (1-t)f + t \cdot \mathrm{id}.$$

$\mathrm{Diff}_\partial D^2 \simeq *$ (Smale)

$\mathrm{Diff}_\partial D^3 \simeq *$ (Hatcher) (Re-proven w/ ricci flow)

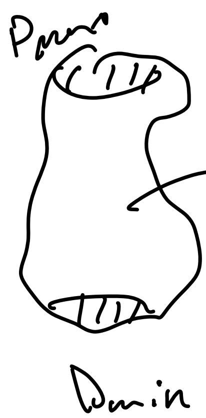
Our First Nontrivial Example / Application.

Exotic Spheres (Milnor):

There are 7-mfds M homeomorphic to S^7, but not diffeo.

Let M be such a manifold

Fact: M will have 2 critical pts.



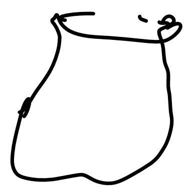
gradient vector field is nonvanishing

Flowing gives diffeo

$$M \setminus \operatorname{int}(D_{min} \sqcup D_{max}) \cong \partial D_{min} \times [0,1]$$

rel ∂D_{min}.

glue this to D_{min} to get big disk



$\} \tilde{D}_{min}$

$$M = D^7 \cup_\varphi D^7$$

$$\varphi : \partial \tilde{D}_{min} = S^6 \to S^6 = \partial D_{max}$$

Extend φ over D_{max} by a homeomorphism,

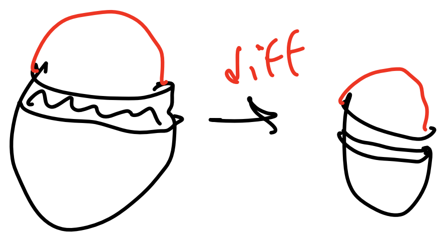
but Alexander trick implies we can instead isotope this to replace

$D^7 \cup_\varphi D^7$ with $D^7 \cup_{id} D^7 = S^7$.

(Reeb's Thm)

By Milnor, φ does not extend by a diffeomorphism over D^7_{max}.

$\Rightarrow$ $\varphi : S^6 \to S^6$ is not isotopic to the identity.



>> Homotopy Theory Enters the Chat

There is a fiber sequence:

$$\underbrace{\mathrm{Diff}_\partial(D^6)}_{D_{min}} \to \mathrm{Diff}^+(S^6) \to \underbrace{\mathrm{Emb}(D^6, S^6)}_{D_{max}}$$

<u>Isotopy extension Thm</u>.

Set-Theoretic Fiber = homotopy fiber.

$\mathrm{Emb}(D^6, S^6) \xrightarrow{\sim} \mathrm{Fr}(TS^6)$

take the derivative.

$\mathrm{Fr}(TS^6) \simeq SO(D+1)$ & is path-connected.

LES says

$\pi_0 \mathrm{Diff}_\partial D^6 \twoheadrightarrow \pi_0 \mathrm{Diff}^+ S^6$

$\Rightarrow \pi_0 \mathrm{Diff}_\partial D^6$ is not trivial.

In fact rotations give inclusion

$SO(D+1) \to \mathrm{Diff}^+(S^d)$

split the fiber sequence so

$\mathrm{Diff}^+(S^d) \simeq \mathrm{Diff}_\partial D^d \times SO(d+1)$

Exotic spheres & Diff (D^m)

Θ_{d+1} denote gp of homotopy spheres of up to diffeo

$+ = \#$.

$\mathrm{Diff}_\partial D^d \longrightarrow \Theta_{d+1}$

Extend diffeo of D^d to S^d by identity & glue two copies of D^{d+1}

(Cerf) $d \geq 5$: $\pi_0 \mathrm{Diff}_\partial (D^d) \cong \Theta_{d+1}$

Reasons:

1) H-cobordism Thm:

Def: H-cobordism

$$M \underset{\partial_0}{\overset{\sim}{\hookrightarrow}} W \underset{\partial_1}{\overset{\sim}{\hookleftarrow}} M'$$

H-cobordism Thm (Smale):

W an H-cobordism of M, M'.
If $\pi_1 W = 0$

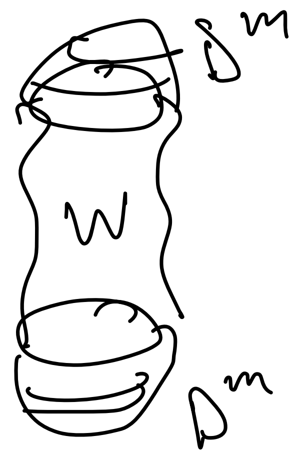
$\Rightarrow$ $W \cong M \times I$ (rel M)

weird:

$f: M \to N$ a smooth map
$f \simeq g$ via F

Thm (Smale, Poincaré Conj, $d \geq 5$)

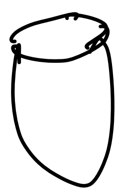
M mfd dim ≥ 6 homotopy equiv to S^m, then it is homeomorphic to S^m.



$\partial D^m \hookrightarrow W$ htpy equiv

(simply connected, homology equiv, Mayer-Vietoris, whitehead)

$W \underset{\text{diffeo}}{\cong} S^{n-1} \times [0,1]$ rel $S^{n-1} \times \{0\}$

 glue by diffeo

$M \cong D^m \cup S^{m-1} \times [0,1] \cup D^m$

$f : S^{m-1} \to W$ glue by diffeo.

Alexander Trick:
extend f to homeomorphism
of D_f^m

Upshot: from the proof:

$$\mathrm{Diff}(S^n) \circlearrowleft \mathrm{Diff}(D^n)$$

measures the diff. between smooth & TOP cats.

More importantly:

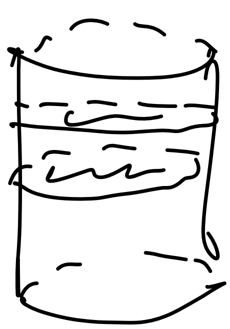
Every homotopy sphere is $D^{n+1} \cup_e D^{n+1}$, e a Diffeo, so

$$\begin{array}{ccc} \pi_0(\mathrm{Diff}\, D^n) & \xrightarrow{\sim} & \pi_0(\mathrm{Diff}\, S^n_+) \\ & & \Big\downarrow \wr \\ & & \Theta_{n+1} \end{array}$$

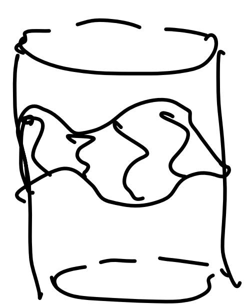
Injectivity requires a subtler notion:

<u>Def:</u> Pseudo-isotopy between f_0, f_1 of M is a diffeo F of $M \times [0,1]$ fixing a neighborhood of $\partial D^d \times [0,1]$ & restricting to f_0, f_1 on ∂.

Isotopy



pseudo:



Cerf: $d \geq 6$ when M is simply connected, pseudo-isotopy implies isotopy (needed for injectivity)

isotopy: $\mathrm{Diff}_\partial(M)$

$\simeq$ pseudoisotopy: $\widetilde{\mathrm{Diff}}(M)$

$$\mathrm{Diff}_\partial M \simeq |\mathrm{Sing}_\bullet(\mathrm{Diff}_\partial M)|$$

$$\begin{tikzcd}
M \times \Delta^r \arrow[rr, "\cong"] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & M \times \Delta^r \arrow[dl] \\
& \Delta^p &
\end{tikzcd}$$

$\mathrm{Map}(\Delta^r, \mathrm{Map}(M, M))$

$\widetilde{\mathrm{Diff}}\, M$ is the same but triangle only commutes up to face maps.

$(\widetilde{\mathrm{Diff}}\, M)_1$:

$$\left.\begin{array}{ccc} M \times I & \xrightarrow{\cong} & M \times I \\ & \searrow \quad * \quad \swarrow & \end{array}\right\} \text{pseudo-isotopy.}$$

What we really want to study is

$B\,\mathrm{Diff}_\partial(D^d)$

$[X, B\mathrm{Diff}_\partial(D^d)] \xleftarrow{\sim} \{\mathrm{Diff}_\partial\, D^d\text{-bundles}\}/\mathrm{iso}$

$$\begin{array}{c} \{E\} \\ \downarrow \\ \{E \times_{\mathrm{Diff}_\partial D^d} D^d\} \end{array} \quad \begin{array}{l} \| \wr \\ \{\text{Smooth } D^d\text{-bundles trivial boundary}\}/\mathrm{iso} \end{array}$$

$\pi_n B\mathrm{Diff}_\partial(D^d) \cong \pi_{n-1}\mathrm{Diff}_\partial D^d$.

who cares about higher homotopy ~~gps~~?

* cohomology $B\mathrm{diff}_\partial(D^d) \Leftrightarrow$ char classes on disk bundles

* $M_\partial(D^n)$ moduli space of n-dim smooth mflds homeomorphic to D^n, standard smooth structure near ∂D^n.

$$M_\partial D^n \simeq \coprod_{\{\sigma\}} B\mathrm{diff}_\partial(D^n_\sigma)$$

*

$$\pi_j(\widetilde{\mathrm{Diff}}(*))$$

$$\|$$

pseudo-isotopy classes of oriented diffeo S^j

$$\cong$$

Θ_{j+1} = smooth $(j+1)$ dim homotopy spheres

Note that Cerf's thm implies that $\mathrm{Diff} \to \widetilde{\mathrm{Diff}}$ is a π_0-iso

Cool Thm:

$$\widetilde{\mathrm{Diff}}(D^d) / \mathrm{Diff}_\partial(D^d) \longrightarrow B\mathrm{diff}(D^d)$$

iso on rational homotopy gps.

$\mathbb{S} \to H\mathbb{Z}$

$A(*) = K(\mathbb{S}) \to K(\Sigma^\infty \Omega B\mathbb{Z}_+) \simeq K(\Sigma^\infty \mathbb{C}_+) \simeq K(\mathbb{Z})$ linearization

$A(*) = \mathbb{S} \times Wh^{\mathrm{Diff}}(*)$

$$\frac{\widetilde{\mathrm{Diff}}\, D^d}{\mathrm{Diff}\, D^d} \longrightarrow \Omega^\infty \Omega Wh^{\mathrm{diff}}(*)$$

stable pseudo-isotopy space

hC_2

$B\mathrm{Diff}(M)$

iso on $i \leq d/3$

$\mathbb{S}$ has finite homotopy gps

$$\Rightarrow \quad \pi_i\, \Omega^\infty\left(\Omega \mathrm{Wh}^{\mathrm{Diff}}(*)_{hC_2}\right) \otimes \mathbb{Q}$$

$$= \begin{cases} K_i(\mathbb{Z}) \otimes \mathbb{Q} & d \text{ odd} \\ 0 & d \text{ even} \end{cases}$$

Sample (serious application)

$$\varphi : \widetilde{\mathrm{Diff}}(*)/\mathrm{Diff}(*)$$

$$\downarrow$$

$$\Omega^\infty \Omega \mathrm{Wh}^{\mathrm{diff}}(*)_{hC_2}$$

$$\pi_j(\mathrm{Diff}(\,\cdot\,)) \simeq \Theta_{j+1}$$

$$\downarrow$$

$$\pi_j K(\mathbb{Z}) \otimes \mathbb{Q}$$

evaluating φ on homotopy gps gives invariants of exotic spheres

& on rational homotopy gps in a range gives invariants valued in $K_i(\mathbb{Z}) \otimes \mathbb{Q}$.