Motivation:

(1) Waldhausen's $S_\bullet$ construction produces a Spectrum in a happier way than the Q-construction.

(2) There are other interesting categories that we want to consider that are not exact.

Topology Example: Fix a space X, and consider the Category $R_f(X)$ of finite retractive spaces over X, i.e.: (Y, X) is homotopy equiv to a relative finite CW pair, with a retraction $r: Y \to X$ such that $r \circ i = id_X$.

Remark: $X = *$, $R_f(X)$ is nothing but finite CW-complexes.

Goal #1: Generalize these examples into a type of category suitable for K-theory, but robust enough to include the examples above.

Waldhausen Categories

<u>Definition 1:</u> Let $\mathcal{C}$ be a category and $0 \in Obj(\mathcal{C})$ that is initial & terminal. Then 0 is a zero object and $\mathcal{C}$ is said to be pointed.

<u>Definition</u> $\mathcal{C}$ is a category with cofibrations if it has a subcategory $co\mathcal{C}$, with morphisms denoted by an arrow $\rightarrowtail$ such that the following properties are satisfied:

1) Every iso is a cofibration
2) $\forall A \in Obj(\mathcal{C})$, $0 \rightarrowtail A$ is a cofib.
3): (cobase change)

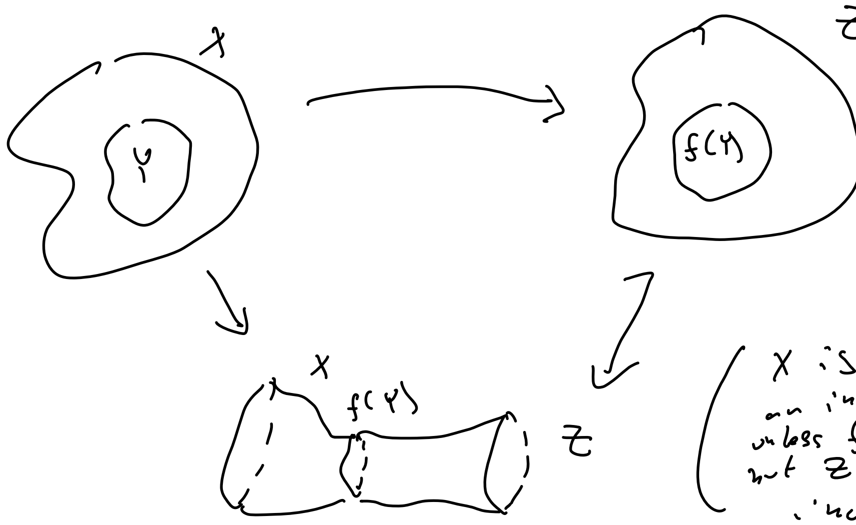
$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & C \arrow[d, tail] \\ B \arrow[r] & B \sqcup_A C \end{tikzcd}$$

exists given $A \rightarrowtail B$ a cofib. Moreover, $C \rightarrowtail B \sqcup_A C$ is a cofib.

Remark: $\mathrm{co}\mathcal{C}$ contains all objects of $\mathcal{C}$ by (1).

5: Cofibrations are modeled on inclusions of CW complexes.

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, hook] & Z \arrow[d, hook] \\ X \arrow[r] & X \cup_Y Z \end{tikzcd}$$



(X is not an inclusion, unless f is, but Z is an inclusion)

<u>Final remark</u>: coproducts exist,

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, tail] & Z \arrow[d, tail] \\ X \arrow[r] & X \sqcup_0 Z \end{tikzcd}$$

<u>Def</u>: An <u>exact functor</u> $F : \mathcal{C} \longrightarrow \mathcal{D}$ between categories w/ cofibrations preserves zeros, cofibrations & pushouts.

<u>Def</u>: $\mathcal{C}$ a cat. w/ cofibrations, then

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & 0 \arrow[d, tail] \\ B \arrow[r] & B \sqcup_A 0 \end{tikzcd}$$

exists for all cofibrations $A \rightarrowtail B$, and we write

$$A \rightarrowtail B \twoheadrightarrow B/A$$

(cofibration sequence).

Definition: A Waldhausen Category is a category with cofib. $\mathcal{C}$ and another subcategory $w\mathcal{C}$ with morphisms $A \xrightarrow{\sim} B$ such that

1: Every iso is a weak equivalence

2: If we have

$$\begin{tikzcd} C \arrow[d, "\sim"] & A \arrow[l] \arrow[r, tail] \arrow[d, "\sim"] & B \arrow[d, "\sim"] \\ C' & A' \arrow[l] \arrow[r, tail] & B' \end{tikzcd}$$

then the induced map $B \sqcup_A C \xrightarrow{\sim} B' \sqcup_{A'} C'$ is a weak equivalence

Remark:

$$\begin{tikzcd} A \arrow[d, "\sim"] & 0 \arrow[l] \arrow[r, tail] \arrow[d, "\sim"] & B \arrow[d, "\sim"] \\ A' & 0 \arrow[l] \arrow[r, tail] & B' \end{tikzcd}$$

$B \sqcup_0 A \xrightarrow{\sim} B' \sqcup_0 A'$

Remark: Sometimes $w\mathcal{C}$ is omitted and is taken to be all isomorphisms.

Example 1: Let $\mathcal{C}$ be an exact Category. Take admissible monomorphisms (appearing in $A \hookrightarrow B \twoheadrightarrow C$) as cofibrations and $w\mathcal{C} = \mathrm{iso}\, \mathcal{C}$.

This is a Waldhausen Category.

→ to see this, embed $\mathcal{C} \hookrightarrow \mathcal{A}$ and show that cofibrations are closed under pushout, i.e.: $0 \to A \to B \to C \to 0$ exact, $A \to D$ any map, then there is a SES

$$0 \to D \to B \sqcup_A D \xrightarrow{\pi} C \to 0$$

(project D to zero, s $\ker(B \to C) = A$)

Before giving more examples, I'd like to give a definition (so that I can say something interesting.)

<u>Definition</u>: Let $\mathcal{C}$ be Waldhausen. Then $K_0(\mathcal{C})$ is the free abelian gp gen. on $[A]$ for $[A] \in ob(\mathcal{C})$, where

1) $A \xrightarrow{\sim} B \Rightarrow [A] = [B]$
2) $A \rightarrowtail B \twoheadrightarrow B/A \Rightarrow [B] = [A] + [B/A]$.

Example 2: $Ch^b_\bullet(\mathcal{E})$ for R a comm. ring.

degree wise mono = cofib.

quasi-iso, iso, etc. = weak equiv

* (Thomason & Trobaugh 1.11.7): R Noetherian

Gillet-Waldhausen: $K(\underset{\text{f.g.}}{P}(R)) = K(Ch^b_\bullet \, P(R))$

$P(R)$: f.g. proj. modules, weak equiv := quasi-iso.

$K_0(A) = K_0(Ch^b(A))$ for any ab. category (works for schemes as well)

proof for K_0 is similar to case R-regular "in spirit."

$K_0(P(R)) \hookrightarrow K_0(Ch^b_\bullet(M(R)))$

$0 \to 0 \to 0 \to [P] \to 0$

$K_0(Ch^b_\bullet P(R)) \longrightarrow K_0(P(R))$

$[A_n \to A_{n-1} \to \cdots \to A_1 \to 0] \mapsto \underbrace{\Sigma(-1)^i[A_i]}_{\text{euler characteristic.}}$

- Not completely vacuous. Recall that if X is f.d., i.e: $\exists\ Y \in R_f(X)$ with $Y \simeq$ finite c.w. complex, then $C_\bullet(\tilde{X}) \in (Ch^b_\bullet[\pi_1 X])$, and the map defined above is the Wall finiteness obstruction.

Example 3: As above, consider $R_f(X)$. Let cofibrations be topological cofibrations in $R_f(X)$.
Weak equivalences can be:
quasi-iso for a homology theory, (weak) homotopy equiv (Rel X).

Take homotopy-equiv:

Then $K_0(R_f(*))$ is such that
finite CW complexes

$$0=[D^k]=[S^{k-1}]+[S^k],$$

so $[S^k]=(-1)^k[S^0]$,
inductively, $K_0(R_f(*))=\mathbb{Z}$, and given a finite CW-complex, say T^2,

cofiber sequence,

so
$$[T^2]=[S^1\vee S^1]+[S^2]$$
$$=-[S^0]-[S^0]+[S^0]=-1\cdot[S^0]$$

reduced Euler characteristic.

Gives map: $K_0(R_f(*))\xrightarrow{\sim}\mathbb{Z}$.

A very similar argument shows that $K_0(R_f(X))=\mathbb{Z}$ for all X, by taking $\tilde{\chi}(Y/X)$.

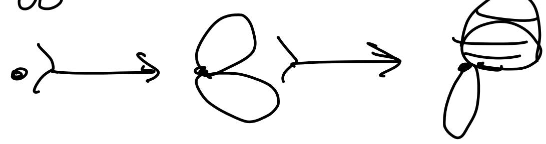
Rmk: $K_0(R_f^{\text{finitely dominated}}(X))=K_0(\mathbb{Z}[\pi_1(X)])$

Higher K-Theory

The idea: Construct a category of the n-filtrations of a space by filtrations:

$$0 = A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n.$$

Analogous to



} Inductively build CW-complex

For a simplicial set we need face maps,

$$S_n \mathcal{C} \to S_{n-1} \mathcal{C},$$

corresponding to $[n-1] \to [n]$. There should be $n+1$ one of them, which is to omit A_j and do

$$\begin{tikzcd} A_{j-1} \arrow[r, tail] \arrow[rr, tail, bend right] & A_j \arrow[r, tail] & A_{j+1} \end{tikzcd}.$$

But, we can't omit A_0, since we want filtrations to start at 0. Instead, omit A_0, and replace the sequence with

$$0 = A_1/A_1 \rightarrowtail A_2/A_1 \rightarrowtail \cdots \rightarrowtail A_n/A_1.$$

But A_n/A_i is only defined up to canonical iso. How do we make it functorial?

Want a face map $S_n\mathcal{C} \to S_{n-1}\mathcal{C}$, so we will ultimately need to make functorial choices for all A_j/A_i $(1 \le i < j \le n)$.

Hence, $\mathrm{Ob}(S_n\mathcal{C})$ will be

$$0 = A_0 \rightarrowtail \cdots \rightarrowtail A_n$$

with <u>choices</u> of quotients A_j/A_i.
In other words:

$$\begin{tikzcd}
0 \arrow[r, tail] & A_{01} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & A_{0n} \arrow[d, two heads] \\
 & 0 = A_{11} \arrow[r, tail] & A_{12} \arrow[r, tail] \cdots & A_{1n} \arrow[d, two heads] \\
 & & 0 = \cdots & \vdots \arrow[d, two heads] \\
 & & & A_{nn}
\end{tikzcd}$$

verticals are quotients.

$A_{ij} = A_j/A_i$,

$$A_{ij} \rightarrowtail A_{ik} \twoheadrightarrow A_{jk}$$

is a <u>cofibration sequence</u>.

<u>Remark</u>: We could take objects of $S_n\mathcal{C}$ to be as above. Then, sending this to n-step filtrations has an inverse operation by "filling in" the diagram w/ choices of subquotients. Lemma 1.1.3 Waldhausen.

We can think of this in another way that can help.

Def: $\mathcal{C}$ a cat. $ar(\mathcal{C})$ has objects morphisms $A \to B$ & morphisms $(A \to B) \to (A' \to B')$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd}$$

Def: $\mathcal{C}$ a cat. w/ cofibrations. Then let $S_n\mathcal{C}$ be the category of functors $A: ar[n] \longrightarrow \mathcal{C}$, $(i \to j) \mapsto A_{ij}$ $\ni$

1) $A_{i \to i} = 0 \ \forall i$

2) for composable arrows the morphism $A_{i \to j} \rightarrowtail A_{i \to k}$ is cofib.

3) for $i \to j$ and $j \to k$,
$A_{i \to j} \rightarrowtail A_{i \to k} \twoheadrightarrow A_{j \to k}$

I would pursue this further, but unraveling the definitions gives the same thing.

(3) is the analog to second iso:
$(A_k/A_i)/(A_j/A_i) \cong A_k/A_j$.

$S_0 \mathcal{C} = 0$
$S_1 \mathcal{C} = \mathcal{C}$

<u>Important Example:</u> $S_2 \mathcal{C}$.

let

$$\begin{tikzcd} 0 \arrow[r, tail] & A_{01} \arrow[r, tail] \arrow[d] & A_{02} \arrow[d] \\ & A_{11} = 0 \arrow[r, tail] & A_{02}/A_{01} \end{tikzcd} \in \mathrm{Obj}(S_2 \mathcal{C}).$$

this is the data of

$$A_{01} \rightarrowtail A_{02} \twoheadrightarrow A_{02}$$

(Since morphisms are based, we know the other subobjects, i.e. 0 are respected.)

$S_2 \mathcal{C}$ is a Waldhausen Category:

$$\begin{tikzcd} A_1 \arrow[r, tail] \arrow[d] & A_2 \arrow[r, two heads] \arrow[d] & A_{12} \arrow[d] \\ A_1' \arrow[r] & A_2' \arrow[r, two heads] & A_{12}' \end{tikzcd}$$

weak equiv if all vertical maps are.
Cofibration if $A_1 \to A_1'$, $A_{12} \to A_{12}'$ are, and
$A_1' \cup_{A_1} A_2 \to A_2'$ are.

then $\partial : S_n \to S_{n-1}\mathcal{C}$ is now functorial

degeneracy maps are maps
maps $\sigma : S_n\mathcal{C} \to S_{n+1}\mathcal{C}$

corresponding to unique nondecreasing map $[n+1] \to [n]$ & compositions of them.

$S_\bullet\mathcal{C}$ is then simplicial object in $\mathcal{C}$,

$$S_\bullet\mathcal{C} : \Delta^{op} \to \mathrm{Cat}$$
$$[n] \to S_n\mathcal{C}.$$

Therefore, $N S_\bullet\mathcal{C}$ will be a bi-simplicial set

Def: A bisimplicial set is a functor
$$F : \Delta^{op} \times \Delta^{op} \to \mathcal{C}$$
$$([n],[m]) \mapsto F_{nm}$$

There are several ways to take the geometric realization.

(1) $\forall m \in \mathbb{N}$, $B_m := |F_{\bullet m}|$

(2) " " " " $|F_{m \bullet}|$

(3) $X_n = |A_{nn}|$.

Surprisingly, all are homeomorphic. we use (3) because it's nice

now, $N_\bullet S_\bullet \mathcal{C} : \Delta^{op} \times \Delta^{op} \longrightarrow Set$

$([n],[m]) \to N_n S_m \mathcal{C}$.

is a bisimplicial set.

If $\mathcal{C}$ was Waldhausen, then $S_n \mathcal{C}$ also gets weak equivalences, namely, an arrow $A \to A'$, $A_{ij} \longrightarrow A'_{ij}$ should be in $\mathcal{C}$ for all $i \le j$.

The cofibrations require that $A_0 \longrightarrow A'_0$

$$\begin{array}{ccccc} A_{ij} & \rightarrowtail & A_{ik} & \twoheadrightarrow & A_{jk} \\ \downarrow & & \downarrow & & \downarrow \\ A'_{ij} & \rightarrowtail & A'_{ik} & \longrightarrow & A'_{jk} \end{array}$$

is a cofibration in $S_2 \mathcal{C}$:

<u>Def:</u> Define the <u>K-theory space</u>

$$K(\mathcal{C}) := \Omega |wS_\bullet \mathcal{C}|$$

& the algebraic K-groups by

$$K_n(\mathcal{C}) := \pi_n(K(\mathcal{C}))$$

ostensibly:

$K : \mathcal{C}_{\mathrm{wald}} \longrightarrow$ Bisimplicial sets.

Actually, the functor is better.

(1) Coproduct gives $K(\mathcal{C})$ the structure of an H-space

(2) A spectrum via delooping

} Jiggle.

" <u>$K_0 = K_0$</u> ", $\pi_0(\Omega wS_\bullet \mathcal{C}) = \pi_1(wS_\bullet \mathcal{C}) = K_0(\mathcal{C})$.

The geometric relization of the bisimplicial set can also be considered as the realization of simplicial space

$$[n] \longmapsto N_\bullet w S_n \mathcal{C} .$$

Since $S_0 w \mathcal{C}$ is trivial category, so it has only one point. But then $|N_\bullet w S_\bullet \mathcal{C}|$ is connected.

Now, note that $|N_\bullet w S_n \mathcal{C}| \times \Delta^n$ are glued along face maps, so in the $n=1$ case, we have a map

$$|N_\bullet w S_1 \mathcal{C}| \times \Delta^1$$
$$\|$$
$$|N_\bullet w \mathcal{C}| \times \Delta^1$$

where $s_0 : 0 \longmapsto (0 \rightarrowtail 0 \twoheadrightarrow 0)$ collapses $0 \times \Delta^1_\bullet$ to a point, and $\underbrace{d_0, d_1}_{\text{face maps}}$ collapse $N_\bullet w \mathcal{C} \times \partial\Delta^1$. reduced suspension

Altogether: $\tilde{\Sigma}|N_\bullet w\mathcal{C}| \longrightarrow |N_\bullet wS_\bullet \mathcal{C}|$.

Then, taking adjoint map

$$|N_\bullet(w\mathcal{C})| \longrightarrow \Omega|wS_\bullet \mathcal{C}|.$$

(Rognes Lemma 8.4.1, Weibel remark 8.3.2)

Here, every object in $\mathcal{C}$ corresponds to an element in $\pi_1|wS_\bullet\mathcal{C}|$.

* ~~key point~~: If they can be joined by a path in $N_\bullet w\mathcal{C}$ (are weakly equiv.) they correspond to the same element of $\pi_1|wS_\bullet\mathcal{C}|$, i.e:

$$\pi_1(|wS_\bullet\mathcal{C}|) \quad \text{and} \quad K_0(\mathcal{C})$$

have the same <u>generators</u>.

Now, $\pi_0|N_\bullet S_2 w\mathcal{C}|$ are equivalence classes of exact sequences, and recall that relations in $\pi_1(|wS_\bullet\mathcal{C}|)$ come from

$\partial_1(x) = \partial_2(x) + \partial_0(x)$ for $x \in \pi_0(|N_\bullet S_2 w\mathcal{C}|)$.

of course, for an exact sequence

$A_1 \rightarrowtail A_2 \twoheadrightarrow A_3$, this is exactly $K_0\mathcal{C}$:)

$A_2 = A_1 \oplus A_3$ $\qquad \mathcal{C} = \mathrm{Mot}_{\mathbb{R}}$

$\{n\} \mapsto |w \quad S_n^{\text{split}} \mathcal{C}| \quad " \overset{?}{=} " \; K(\mathrm{Proj})$

$S_2^{\text{split}} \mathcal{C}$