The Freudenthal Suspension Theorem

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Overview

- The higher homotopy groups π_i turn topological problems into algebraic ones,
  BUT π_i is notoriously difficult to compute
  - ↳ no van Kampen (π_1)
  - ↳ no excision (homology)

- For example: $\pi_i(S^n)$ ⟵ simple space = ☺?
  - ↳ trivial, $i < n$
  - ↳ $\mathbb{Z}$, $i = n$
  - ↳ ??, $i > n$ (but see Hatcher p.339)

- The Freudenthal Suspension Theorem says

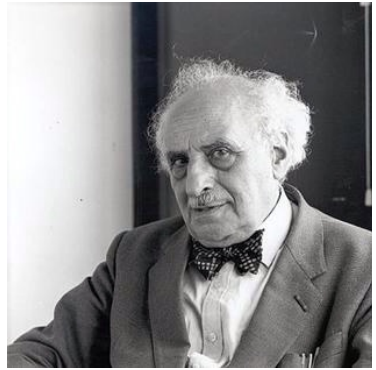
$$\pi_i(X) \cong \pi_{i+1}(\Sigma X)$$

↑ reduced suspension

  for a certain range of i

- enter: stable homotopy theory!

Hans Freudenthal (Sep 17, 1905 – Oct 13, 1990)



(image from Wikipedia)

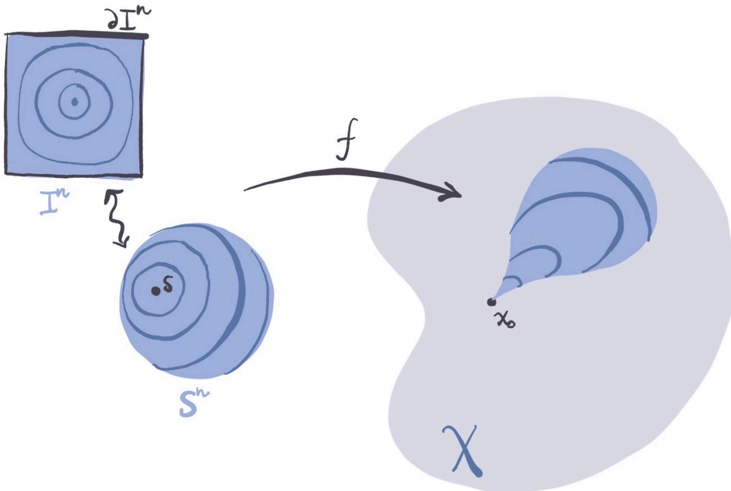
- PhD student of Hopf (1931) in Berlin
- assistant to Brouwer in Amsterdam
- proved FST in 1937 (for spheres)
- persecuted when Nazis invade Amsterdam (1940 – 1945)
- in addition to AT, Freudenthal worked in math history, math education, and literature

Goal: Prove the FST (derive BM)

Key Ingredients

1. Higher homotopy groups
2. Homotopy Excision
3. Suspensions

1. Higher Homotopy Groups

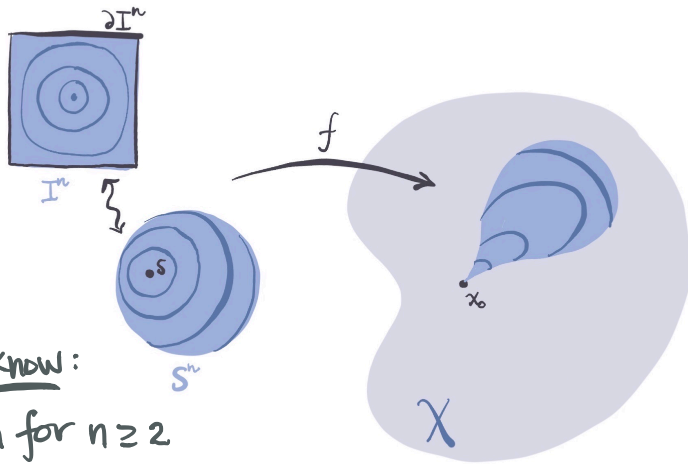


Defn - The n^{th} homotopy group of (X, x_0) is

$$\pi_n(X, x_0) = \{ [f] \mid f : (S^n, s) \to (X, x_0) \}.$$

$\pi_n(X)$ $\quad$ $(I^n, \partial I^n)$

1. Higher Homotopy Groups



Some things to know:

- π_n is Abelian for $n \geq 2$

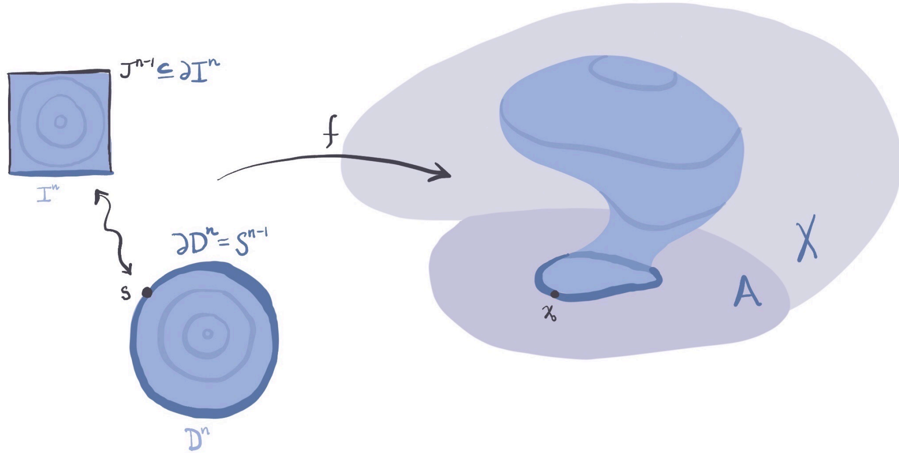
- X is <u>n-connected</u> if $\pi_i(X) = 0$ for all $i \leq n$

- $\pi_n : Top_* \longrightarrow \overset{Ab}{Gp}$ is a functor

  ↳ $\phi : X \to Y$ induces $\phi_* : \pi_n(X) \longrightarrow \pi_n(Y)$

  $[f] \longmapsto [\phi \circ f]$

Relative Homotopy Groups

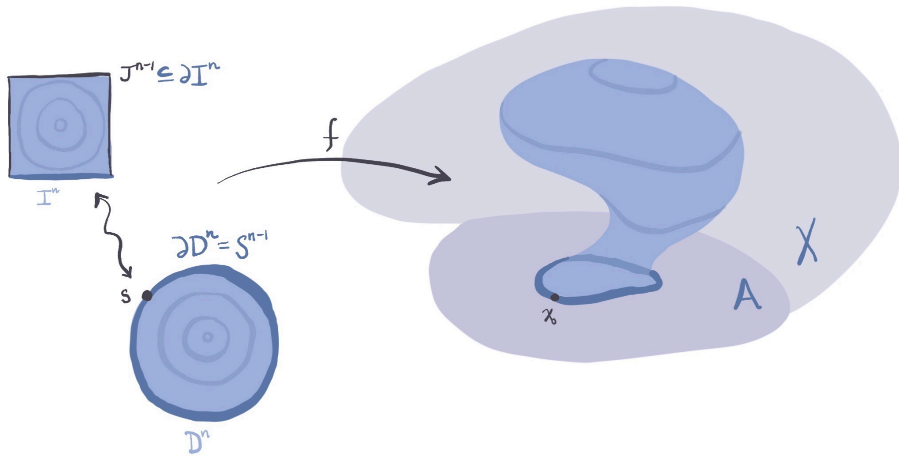


Defn - Let $A \subseteq X$ and $x_0 \in A$. The n^{th} relative homotopy group is

$$\pi_n(X, A, x_0) = \{ [f] \mid f : (D^n, S^{n-1}, s) \to (X, A, x_0) \}.$$

$\pi_n(X, A)$ $\quad$ $(I^n, \partial I^n, J^{n-1})$

Relative Homotopy Groups



Long Exact Sequence

$$\cdots \longrightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial : f \mapsto f|_{S^{n-1}}} \pi_{n-1}(A) \longrightarrow \cdots$$

$i : A \hookrightarrow X \qquad j : (X, x_0) \hookrightarrow (X, A)$

Long Exact Sequence

$$\cdots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X,A) \xrightarrow{\partial} \pi_{n-1}(A) \to \cdots$$

$\pi_i(X,A) = 0 \ \forall i \leq n$

Defn - If (X,A) is n-connected, then i_* is an isomorphism for $i<n$ and a surjection for $i=n$. The inclusion $i: A \hookrightarrow X$ is called an <u>n-equivalence</u>.

n-connected

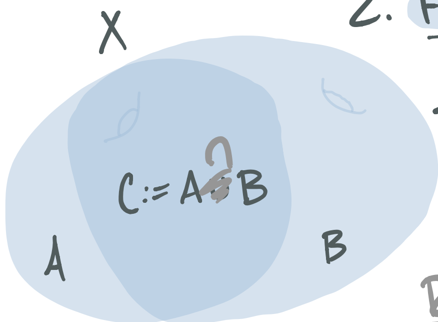
Example $S^n = \partial D^{n+1} \hookrightarrow D^{n+1}$

know: $\pi_i(D^{n+1}) = 0 \ \forall i$
$\pi_i(S^n) = 0 \quad i<n$
$\pi_n(S^n) = \mathbb{Z} \quad i=n$

$\Rightarrow \pi_i(D^{n+1}) \cong \pi_i(S^n) \quad i<n$
$i: \mathbb{Z} \to 0 \quad i=n$

$\rightsquigarrow$ n-equivalence

2. Homotopy Excision



Defn - An <u>excisive triad</u> is $(X; A, B)$ s.t. $A, B \subseteq X$ and $X = A^\circ \cup B^\circ$.

<u>Rmk</u> In homology,

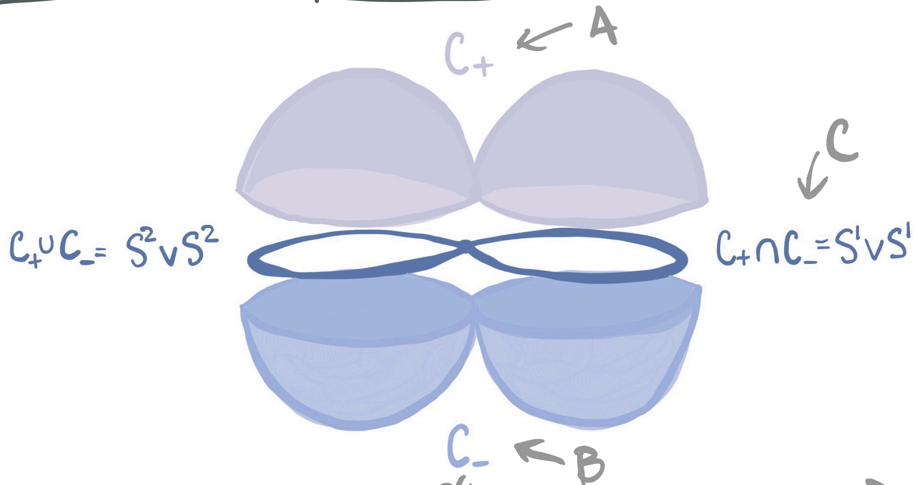
$$(A, C) \hookrightarrow (X, B)$$

induces an isom. on homology

Excision does <u>not</u> hold, in general, for htpy gps

2. Homotopy Excision

Example of Failure of Excision: $S^2 \vee S^2$



Claim $\pi_2(C_+, S^1 \vee S^1) \overset{\not\cong}{\longrightarrow} \pi_2(S^2 \vee S^2, C_-)$

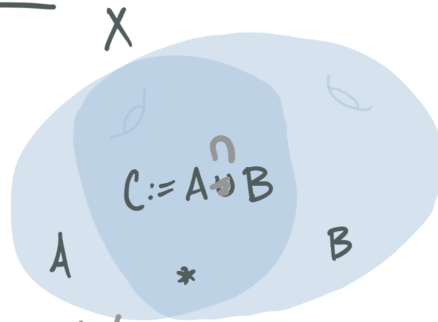
$i=2$

$$\text{not Ab} \rightarrow \underset{\cong}{\pi_1(S^1 \vee S^1)} \quad \not\cong \quad \underset{\cong}{\pi_2(S^2 \vee S^2)} \leftarrow \text{Ab}$$

2. Homotopy Excision

Q. When does excision hold?

A. In a range of dimensions



Thm (Blakers-Massey).

$C \neq \emptyset$

Let $(X; A, B)$ be an excisive triad s.t.

for all $* \in C$

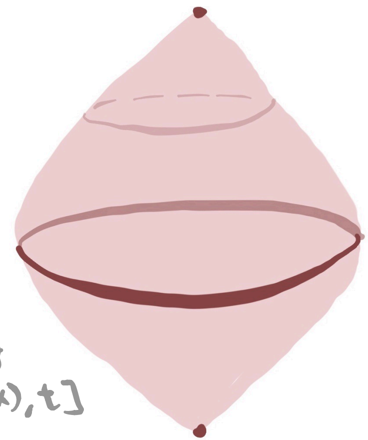
(A, C) is n-connected,

(B, C) is m-connected.

Then $\pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism for $i < n+m$ and a surjection for $i = n+m$. i.e. $(n+m)$-equivalence

Pf idea / Reduce to simpler case (see notes)

3. Suspensions



Defn - The Suspension functor

$$\begin{array}{ccc} S: \mathbf{Top} & \longrightarrow & \mathbf{Top} \\ X & \longmapsto & SX = X \times I \Big/ {X \times \{0\} \atop X \times \{1\}} \\ f \downarrow & \longmapsto & \downarrow Sf : [x,t] \mapsto [f(x), t] \\ Y & \longmapsto & SY \end{array}$$

$S(S^1) \cong S^2$

$S(S^n) \cong S^{n+1}$

$\rightsquigarrow$ $x_0 \in X$, the reduced suspension,

$$\Sigma X := SX / x_0 \times I$$

$\hookleftarrow$ basept

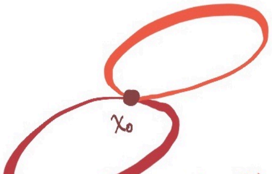
is a functor $\Sigma : \mathbf{Top}_* \longrightarrow \mathbf{Top}_*$

3. Suspensions

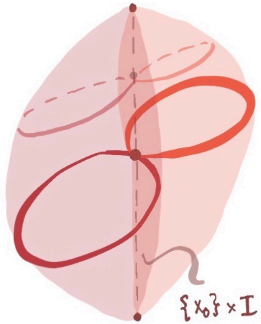
Suspension v.s. reduced suspension ?

SX $\qquad\qquad$ $\Sigma X = SX / x_0 \times I$

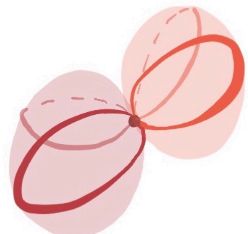
Example $S^1 \vee S^1$



$S^1 \vee S^1$



$S(S^1 \vee S^1)$



$\Sigma(S^1 \vee S^1) \cong S^2 \vee S^2$

$\Sigma(X \vee Y) \cong \Sigma X \vee \Sigma Y$

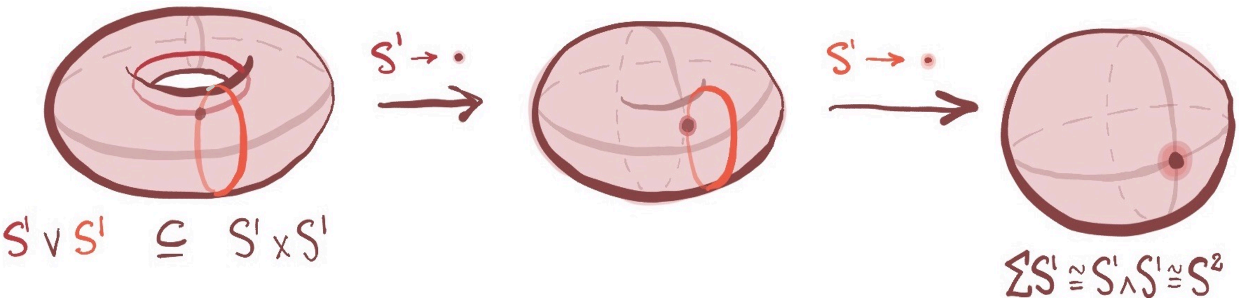
Rmk. If X is CW cpx, then $\Sigma X \simeq SX$.

3. Suspensions

Prop - $\Sigma X \cong X \wedge S^1 = X \times S^1 / X \vee S^1$

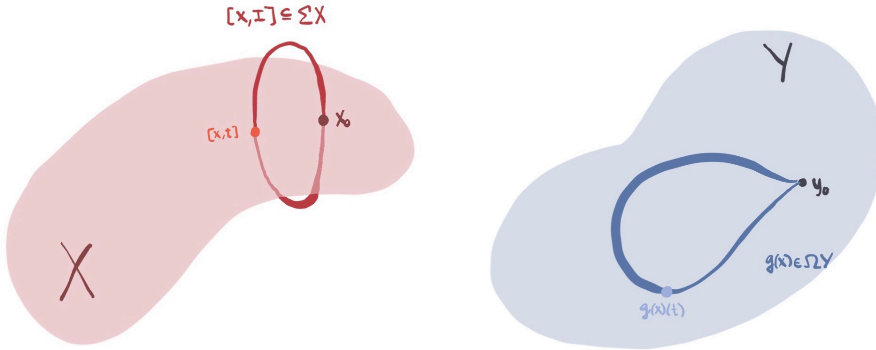
$$\Sigma X = X \times I \Big/ \begin{matrix} X \times \{0,1\} \\ x_0 \times I \end{matrix} = X \times I \Big/ \begin{matrix} X \times \partial I \\ x_0 \times I \end{matrix} \cong X \times S^1 / X \vee S^1$$

Example $\Sigma S^1 \cong S^1 \wedge S^1$ In general, $\Sigma S^n \cong S^{n+1}$



3. Suspensions

Rmk. $\Sigma \dashv \Omega$ i.e. $\mathrm{Top}_*(\Sigma X, Y) \cong \mathrm{Top}_*(X, \Omega Y)$



$f: \Sigma X \to Y$ $\qquad\qquad$ $g: X \to \Omega Y$

$[x,t] \mapsto f[x,t] \quad \rightsquigarrow \quad x \mapsto (t \mapsto f[x,t])$

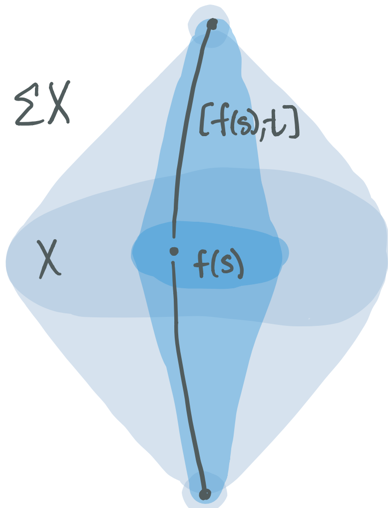
$[x,t] \mapsto g(x)(t) \quad \leftsquigarrow \quad x \mapsto g(x)$

Note $\pi_n(\Omega Y) \cong \pi_{n+1}(Y)$

$S^n \to \Omega Y \quad \leftrightsquigarrow \quad \Sigma S^n \cong S^{n+1} \to Y$

3. Suspensions

Defn - The suspension homomorphism is



$$\Sigma_*: \pi_i(X) \to \pi_{i+1}(\Sigma X)$$

$$[f] \mapsto [\Sigma f]$$

$$\|$$

$$f \wedge \mathrm{id}_{S^1} : S^{i+1} \to \Sigma X$$

$$[s,t] \mapsto [f(s),t]$$

Rmk. Σ_* is the unit of $\Sigma \dashv \Omega$.

nat'l transf $\mathrm{id}_{Top_*} \to \Omega \circ \Sigma$

(see notes)

Thm - If X is $(n-1)$-connected, then $\Sigma_*: \pi_{i-1}(X) \to \pi_i(\Sigma X)$ is an isom. for $i < 2n$ and a surj. for $i = 2n$.

The Freudenthal Suspension Theorem

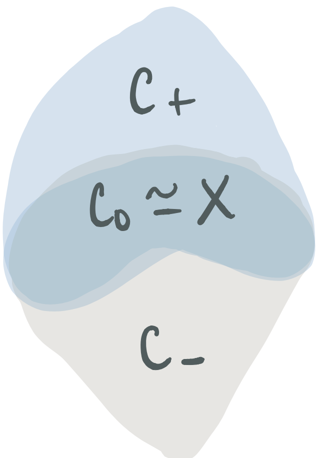
Thm – If X is $(n-1)$-connected, then $\Sigma_* : \pi_{i-1}(X) \to \pi_i(\Sigma X)$ is an isom. for $i < 2n$ and a surj. for $i = 2n$.

Pf Sketch/ Choose a nice excisive cover of ΣX:

C_+ = (reduced) cone over X

C_- = (reduced) cone under X

$C_0 := C_+ \cap C_- \simeq X$



Then

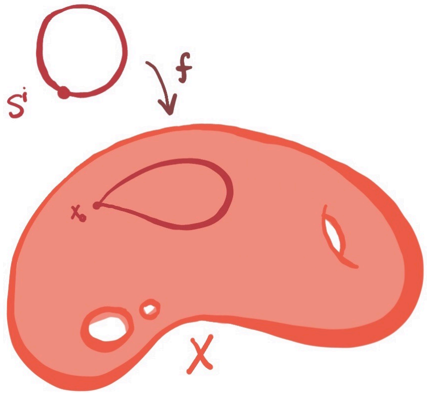
apply BM: 2n-equiv

$$\begin{tikzcd} \pi_i(C_+, C_0) \arrow[r, "i_*"] \arrow[d, "\partial"', "\cong"] & \pi_i(\Sigma X, C_-) \\ \pi_{i-1}(X) \arrow[r, dotted] & \pi_i(\Sigma X) \arrow[u, "\cong", "j_*"'] \end{tikzcd}$$

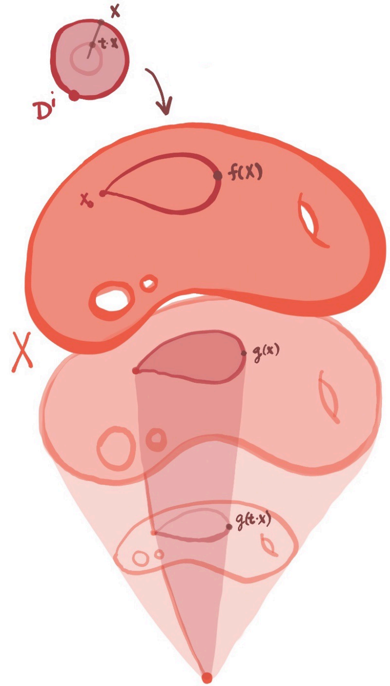
Σ isom $i<2n$
surj $i=2n$, is Σ_*?

The Freudenthal Suspension Theorem

$$\begin{tikzcd}
{[g]} \quad \pi_i(C_-, C_0) \arrow[r, "i_*"] \arrow[d, "\partial"', "\cong"] & \pi_i(\Sigma X, C_+) \\
{[f]} \quad \pi_{i-1}(X) \arrow[r, dashed, "\Sigma_*?"'] \arrow[u, bend left=60, mapsto, "\partial^{-1}"] & \pi_i(\Sigma X) \arrow[u, "j_*"', "\cong"]
\end{tikzcd}$$

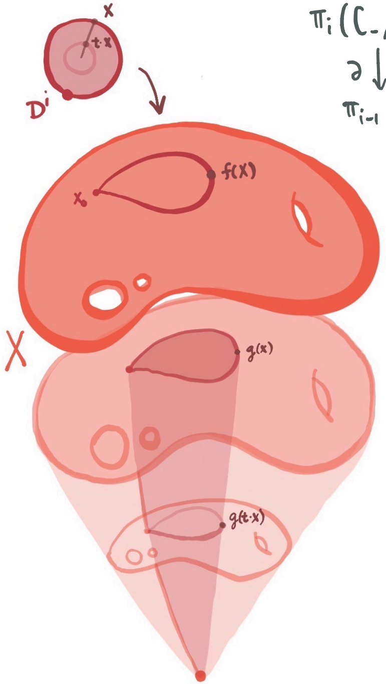


$$\xmapsto{\partial^{-1}}$$



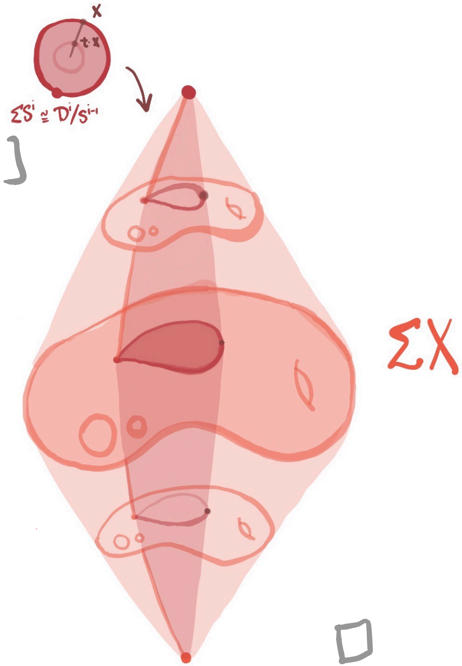
The Freudenthal Suspension Theorem

$$\begin{tikzcd}
{[g]} \arrow[r, mapsto] & {[g] = [\hat{g}]} \arrow[dd, bend left] \\
\pi_i(C_-, C_0) \arrow[r, "i_*"] \arrow[d, "\partial"', "\cong"] & \pi_i(\Sigma X, C_+) \\
\pi_{i-1}(X) \arrow[r, dashed, "\Sigma_*?"'] & \pi_i(\Sigma X) \arrow[u, "\cong"', "j_*"] \\
 & {[\Sigma f]}
\end{tikzcd}$$



i_*

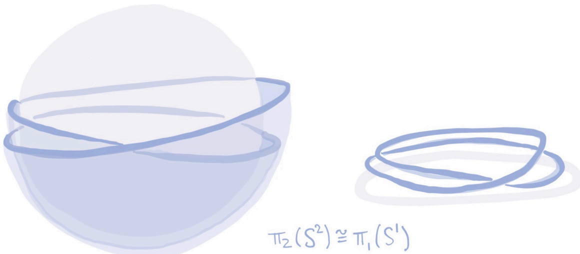
j_*



$\square$

Conclusion: Some Applications

Spheres: FST says $\Sigma_*: \pi_i(S^n) \cong \pi_{i+1}(S^{n+1})$ for $i < 2n-1$



$\pi_2(S^2) \cong \pi_1(S^1)$

$\pi_2(S^2) \cong \mathbb{Z}$

$\Rightarrow \pi_3(S^3) \cong \mathbb{Z}$

$\Rightarrow \dots \; \pi_n(S^n) \cong \mathbb{Z}$

Stable Homotopy Groups: $\Sigma_*: \pi_i(\Sigma^n X) \cong \pi_{i+1}(\underbrace{\Sigma^{n+1} X}_{\Sigma(\Sigma^{n-1}X)})$ for $i < 2n-1$

This means, for fixed i, the maps in

$$\pi_i(X) \to \pi_{i+1}(\Sigma X) \to \pi_{i+2}(\Sigma^2 X) \to \dots$$

become isom. The eventual value is i^{th} stable htpy gp of X,

$$\pi_i^S(X) = \operatorname{colim}_n(\pi_{i+n}(\Sigma^n X)).$$

$\pi_0^S(S^0) \cong \mathbb{Z}$