Infinite Loop Spaces

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MOTIVATION

Given a t.s. X w/ operation $\mu: X \times X \longrightarrow X$, can ask about algebraic properties of μ

Requiring strict assoc., comm., unital mult. is too restrictive! **Thm** Every top. Ab. monoid is product of EM space

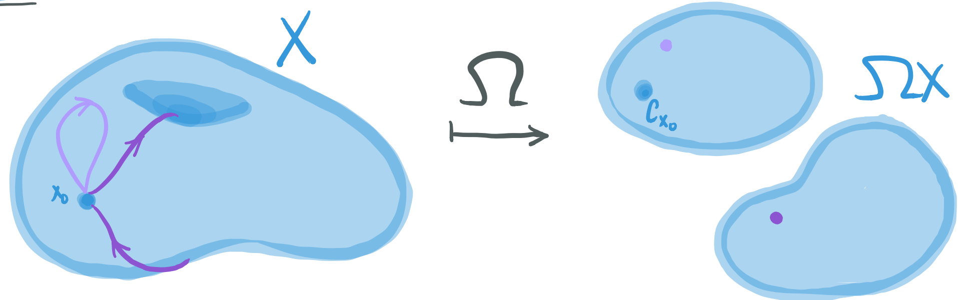
Loop spaces are nice examples where these properties hold up to coherent higher htpy "next best thing"

OVERVIEW

- Loop spaces ΩX
  - ↳ keep going … $\Omega^k X$ …
  - ↳ $\Omega^\infty X$ and Ω-spectra
- Recognition thms
  - ↳ via Operads (May)
    - iterated loop space $\Leftrightarrow$ E_n-algebra
  - ↳ via Segal's Γ-spaces
    - symmetric monoidal category $\longmapsto$ infinite loop space
- Deloopings + Uniqueness (briefly)

Goal: Give introduction to these ideas + convince you they are cool / interesting

LOOP SPACES



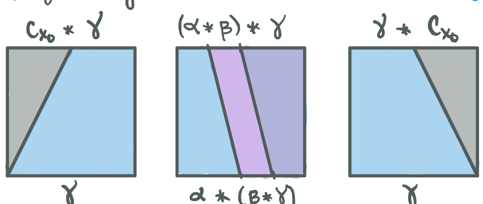
Defn - The <u>loop space</u> functor $\Omega : Top_* \to Top_*$ sends (X, x_0) to

$$\Omega X = Map_*(S^1, X)$$

w/ basepoint c_{x_0}. A map $(X, x_0) \xrightarrow{f} (Y, y_0)$ is sent to $f_* : \phi \mapsto f \circ \phi$.

Rmks - $\pi_n(\Omega X) \cong \pi_{n+1}(X)$ $\quad$ $n=0$ means ΩX is <u>grouplike</u> (π_0 is a gp)

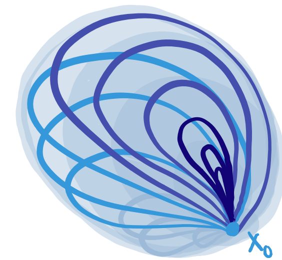
H-space structure : μ given by concatenation $\quad$ unital and assoc? up to htpy



ITERATED LOOP SPACES

Defn - The k^{th} loop space of X is $\Omega^k(X) = \Omega(\Omega^{k-1}(X)) = \mathrm{Map}_*(S^k, X)$.

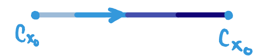
Rmk $\pi_0(\Omega^k X) \cong \pi_k(X)$



map $S^2 \longrightarrow X$

$\Updownarrow$

loop of loops



Defn - Say X is an infinite loop space if there is a sequence $\underbrace{X = X_0, X_1, X_2, \ldots}_{\text{delooping of } X}$ w/ weak equivalences

$$X_n \xrightarrow{\sim} \Omega X_{n+1}$$

By adjunction, get map $\Sigma X_n \longrightarrow X_{n+1}$. $\longleftarrow$ Spectrum!

Recall: A spectrum E is an Ω-spectrum if adjoints of structure maps are weak equivalences

So An infinite loop space is the 0^{th} term of Ω-spectrum

$\rightsquigarrow \Omega^\infty$ is functor from spectra to spaces

$\hookrightarrow$ e.g. $\pi_k(\Omega^\infty \mathbb{S}) = \pi_k^S$

Note: By construction, the htpy gps of $\Omega^\infty E$ are the htpy gps of spectrum E

Ex - Eilenberg-MacLane spaces: $\{K(G,n)\}$ forms Ω-spectrum

$$\pi_k(\Omega K(G,n)) \cong \pi_{k+1}(K(G,n)) = \begin{cases} G & k+1 = n \\ 0 & \text{else} \end{cases}$$

$$K(G, n-1) \xrightarrow{\cong} \Omega K(G,n)$$

RECOGNITION THM (via Operads)

Question: When is a space Y (weakly) equivalent to ΩX for some X? $\Omega^k X$? $\Omega^\infty X$??

Look at necessary conditions: (for ΩX)

ΩX is a group-like H-space
(i) (ii)

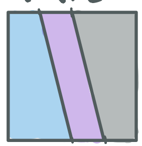
(i) π_0 is a gp

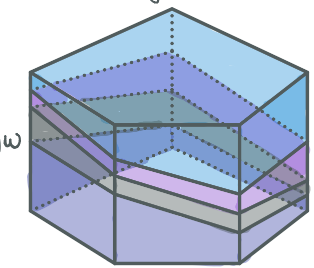
(ii) μ which assoc/unital up to htpy

Is every group-like H-space a loop space?

A. No. Need "coherent higher htpy"

"A_∞-spaces"





htpy coherence for associativity of product x y z w

A_∞- SPACES

Thm (Stasheff) - Let (X, x_0) be a group-like H-space. Then X is a loop space if and only if X is an A_∞-space.

What is an A_∞-space? multiplication is associative up to "coherent higher htpy"

$n=2$: $x \cdot y$ $\rightsquigarrow$ K(0)

$n=3$: $(x\cdot y)\cdot z$ or $x\cdot(y\cdot z)$ $\rightsquigarrow$ K(1)

$n=4$: $(x\cdot y)(z\cdot w)$ — $x\cdot(y\cdot(z\cdot w))$

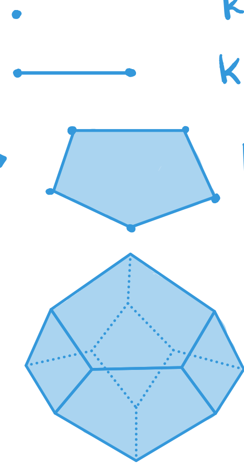
$((x\cdot y)\cdot z)\cdot w$ $\quad$ $x\cdot((y\cdot z)\cdot w)$

or $(x\cdot(y\cdot z))\cdot w$

$\rightsquigarrow$ K(2)

$n=5$: $\rightsquigarrow$ K(3)

⋮ $\quad$ ⋮



Rmk - When multiplication is strictly associative: $*$

Next best thing: contractible, "A_∞- Space"

Formally, this idea is encoded in an operad.

OPERADS (briefly)

Idea: encode n-ary operations
↪ can gen'lize to symmetric monoidal category

Defn (in spaces) An *operad* consists of data:

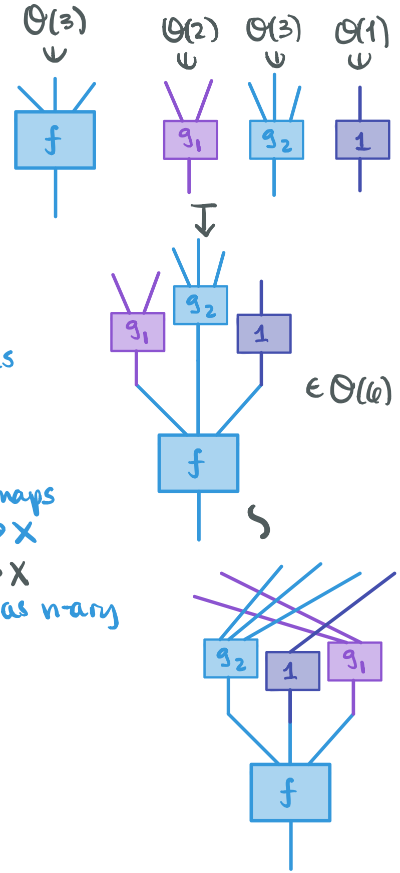
- Space $\mathcal{O}(n)$ $\forall n \geq 1$ s.t.
  - $\mathcal{O}(0) = *$ picks out unit
  - $1 \in \mathcal{O}(1)$ is identity
  - action of Σ_n on $\mathcal{O}(n)$ permuting the variables
- Structure maps $\mathcal{O}(n) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_n) \to \mathcal{O}(k_1 + \cdots + k_n)$

A *morphism* of operads is a collection of maps $\mathcal{O}(n) \to \tilde{\mathcal{O}}(n)$ satisfying some coherence conditions. e.g. comm. w/ structure maps

An *algebra* X over an operad is a collection of maps $\mathcal{O}(n) \times X^n \to X$ ("$f : X^n \to X$") satisfying some coherence conditions. Interpret elmts of $\mathcal{O}(n)$ as n-ary operation on X

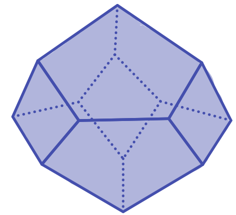
e.g. 1) associative operad $\mathrm{Assoc}(n) = *$ w/ trivial action
⇒ algebras are top. monoid

2) commutative operad $\mathrm{Comm}(n) = *$ w/ free action
⇒ algebras are Ab. top. monoids



Recognition Thm via Operads

Back to A_∞-spaces... Reinterpret: X is A_∞-space $\Leftrightarrow$ X is algebra over K



$K(3)$

Rmk Have notion of A_n spaces   action of $\{K(i)\}_{i \leq n}$

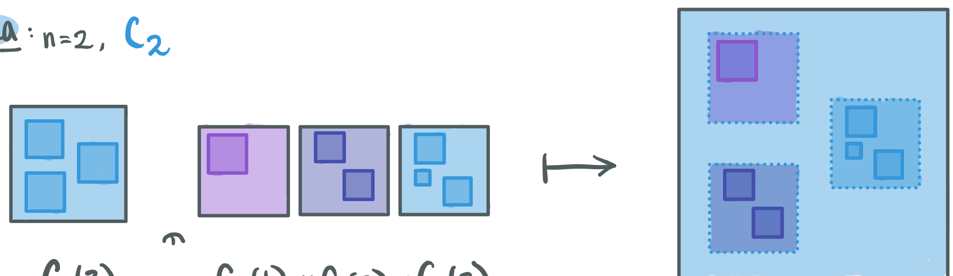
X has strictly assoc. mult. $\Rightarrow$ X is A_∞ $\Rightarrow \cdots \Rightarrow$ X is A_n $\Rightarrow \cdots \Rightarrow$ X is A_1 (H-space)

Thm A group-like space X is a loop space iff it is A_∞ $\leftarrow$ algebra over K

Q. Is there an operad whose algebras are the iterated (infinite) loop spaces?

Yes! n-cubitos C_n

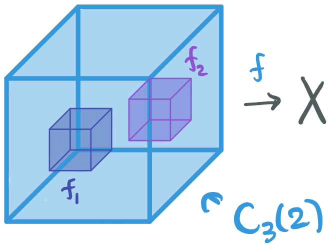
Idea: $n=2$, C_2



$C_2(3) \times C_2(1) \times C_2(2) \times C_2(3) \longmapsto \;\in C_2(6)$

E_∞- SPACES

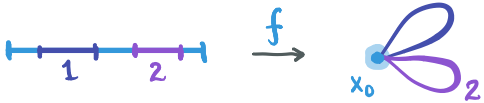
The n-cubitos operad acts on $\Omega^n X = \mathrm{Map}_*(S^n, X) = \mathrm{Map}((I^n, \partial I^n), (X, x_0))$



Given n-cubes $c_1, \dots, c_k$ and maps $f_i : (I^n, \partial I^n) \to (X, x_0)$ for $1 \le i \le k$, construct map $f : (I^n, \partial I^n) \to (X, x_0)$

$$f(t) = \begin{cases} f_i(t) & t \in c_i \\ x_0 & \text{else} \end{cases}$$

e.g. $n=1$:



concatenation of loops

Rmk. An algebra over $\overset{\infty}{n}$-cubitos is $\overset{E_\infty}{E_n}$-space.

Encodes commutativity of mult. up to "coherent higher htpy"

Thm (May) - For $1 \le n \le \infty$, a grouplike space X is an n-loop space $\Leftrightarrow$ X is E_n-space $\leftarrow$ algebra over n-cubitos

Q. What are A_∞-objects, E_∞-objects in other symm. monoidal categories?

Another Approach: Segal's Γ-Spaces

Motivation: construct infinite loop spaces from symm. monoidal categories

$$\text{s.m. Cats} \Rightarrow \Gamma\text{-spaces} \Rightarrow \Omega^\infty \text{ spaces} \Leftrightarrow \Omega\text{-spectra}$$

Defn - Let Γ be the category s.t. Γ^{op} is the category of finite ptd sets + basept preserving maps.

Rmk - Can think of object of Γ as $n^+ := \{0, 1, 2, \dots, n\}$ for some $n \geq 0$.

*

Defn - A Γ-space X is a functor $X : \Gamma^{op} \to Top_*$ s.t. $X(0^+)$ is contractible.
Say X is "special" if $X(S \amalg T) \xrightarrow{\simeq} X(S) \times X(T)$, and "very special" if $\pi_0 X(1^+)$ is a group.

$$X(n^+) \xrightarrow{\sim} X(1^+)^n$$

$\rightsquigarrow$ "only need" $X(1^+)$

Thm (Segal) - Let X be a very special Γ-space. Then $X(1^+)$ is an infinite loop space.

Note: Thm doesn't require "very" b/c can always use group completion

SEGAL'S Γ-SPACES

Thm – Let X be a very special Γ-space. Then $X(1^+)$ is an infinite loop space.

Generalize: Given sym. mon. cat. $\mathcal{C}$, get special Γ-category

↳ functor $\hat{\mathcal{C}} : \Gamma^{op} \to \mathrm{Cat}$ s.t. $\hat{\mathcal{C}}(0^+)$ equiv to $\bullet \circlearrowleft \mathrm{id}$ and $\hat{\mathcal{C}}(n^+) \simeq \mathcal{C}^n$

↳ composition $\Gamma^{op} \xrightarrow{\hat{\mathcal{C}}} \mathrm{Cat} \xrightarrow{\iota} \mathrm{Cat} \xrightarrow{B} \mathrm{Top}_*$ is special Γ-space

↳ Thm says $B\hat{\mathcal{C}}(1^+) = B\mathcal{C}$ "is" ∞-loop space (after gp completion)

Examples

1) $\mathcal{C} = (\iota\mathrm{Fin}, \amalg, \emptyset)$

$\rightsquigarrow \mathbb{S}$ "Priddy-Quillen Theorem"

2) $\mathcal{C} = (\iota\mathrm{Mod}_R(\text{f.g. proj}), \oplus, (0))$

$\rightsquigarrow$ alg. K-theory spectrum of R

3) $\mathcal{C} = (\iota\mathrm{Vect}_{\mathbb{C}}, \oplus, (0))$ enriched over Top

$\rightsquigarrow \mathbb{Z} \times BU$ (top. K-theory)

DELOOPING MACHINE(S)

Idea: Given loop space X, how can we "deloop" it?

Segal's idea: Iterate classifying space for very special Γ-space X, get spectrum BX w/ $BX_n = B^n X$

$$B^n X \xrightarrow{\simeq} \Omega B^{n+1} X.$$

May's idea: use monads + algebras over monads + two sided simplicial bar construction

$$X \xrightarrow{\simeq} \Omega^k B(\Sigma^k, \Sigma^k \Omega^k, X).$$

+ Others...

Thm (May-Thomason, 1978) - For any loop space machine E, there is a nat'l equiv. of spectra $EX \sim BX$

References

J.F. Adams Infinite Loop Spaces

A.M. Osorno "Espacios de lazos infinitos"

E. Belmont "A quick introduction to operads"

D. Freed "Γ-spaces and deloopings"