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The Magic of K-theory
Pt. I

An Introduction to Topological K-theory

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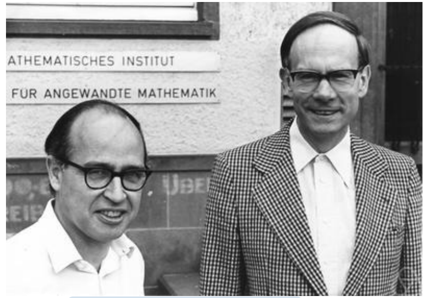
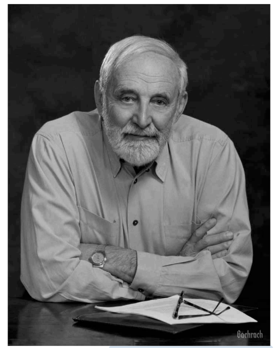
THE IDEAS OF K-THEORY ...

- Study "nice" spaces by studying vector bundles over them
  ↳ compact Hausdorff
  - ↳ $K(X)$ is ring of isom. classes of vec. bundles
    $\oplus \rightsquigarrow +$
    $\otimes \rightsquigarrow \cdot$
  - ↳ $X \mapsto K(X)$ is functorial + represented by $BU \times \mathbb{Z}$
    i.e. $K(X) \cong [X_+, BU \times \mathbb{Z}]_*$
- important thm!! Bott Periodicity (1957)
  - ↳ helps compute $K(X)$ sometimes
  - ↳ extends K to generalized cohomology theory
    $\leftrightsquigarrow$ Ω-spectrum

References: Hatcher's VBKT
May's Concise Course
+ others ... see write-up



M. Atiyah (1929-2019)
& F. Hirzebruch (1927-2012)
from AMS



R. Bott
(1923-2005)
from NYT

BRIEF REVIEW OF COMPLEX VECTOR BUNDLES...

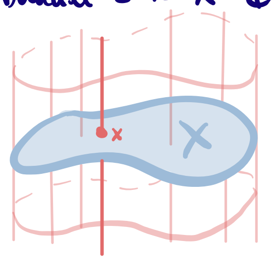
Defn - A <u>vector bundle</u> over X is $E \xrightarrow{p} X$ s.t.

(i) fibers $p^{-1}(x)$ have $\mathbb{C}$-v.s. structure

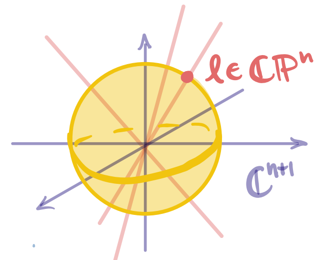
(ii) E is locally trivial

Exs:

trivial bundle $\varepsilon^n := X \times \mathbb{C}^n$



canonical line bundle $H := \{(\ell, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \in \ell\}$ (over $\mathbb{CP}^n$)



Recall Can "add" and "multiply" v.b. using $\oplus$ and $\otimes$
is associative, commutative, distributive up to nat'l isom.

Idea: turn into ring operations

HOW DO WE GIVE VEC. BUNDLES A RING STRUCTURE? for fixed X

① Work with isomorphism classes: Denote by $\mathrm{Vect}^{iso}(X)$

↳ Then properties of $\oplus$ give commutative monoid structure (gp w/out inverses)

② turn into group "formally adjoin inverses"

Defn – (Univ Prop. of Gp Completion) A gp $G(M)$ is group completion of monoid M if:
if $M \xrightarrow{f} A$ for gp A then

$$\begin{tikzcd} M \arrow[r, hook, "i"] \arrow[dr, "f"'] & G(M) \arrow[d, dashed, "\hat{f}"] \\ & A \end{tikzcd}$$

Explicit construction: Grothendieck group

$$Gr(M) = \text{free gp}\{[m] : m \in M\} / [m+n]-[m]-[n]$$

inclusion $M \hookrightarrow Gr(M)$ by $m \mapsto [m]$. Inverse of $[m]$ is $-[m]$

e.g. $G(\mathbb{N}) = \mathcal{F}\{[n] : n \in \mathbb{N}\} / [n+m]-[n]-[m] \cong \mathbb{Z}$

Rmk. If M semi-ring, then $Gr(M)$ is ring. $\mathrm{Vect}^{iso}(X)$ is semiring: $\oplus$ $\otimes$

HOW DO WE GIVE VEC. BUNDLES A RING STRUCTURE? (cont.)

Defn The topological K-theory of X is the ring $Gr(Vect^{iso}(X))$, i.e.

$$K(X) := \text{free gp}\{\cong\text{-classes of v.b.}\} / [E \oplus E'] - [E] - [E'].$$

- an element is "virtual bundle" $[E]-[E']$. not nec. unique
- zero class is $[E]-[E]$
- inverse of $[E]-[E']$ is $[E']-[E]$
- Write $n = [\varepsilon^n]$. Every elmt has representation $[E]-n$.

  $\hookrightarrow$ $E-n = E'-m \iff n=m,\ E \oplus \varepsilon^m \cong E' \oplus \varepsilon^n$

- ring structure: $([E]-n)+([E']-m) = [E \oplus E'] - (n+m)$

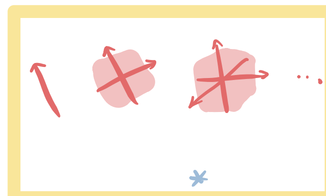
  $([E]-n)\cdot([E']-m) = [E \otimes E'] - n[E'] - m[E'] + (n+m)$

Ex. $K(*)$

① $Vect^{iso}(*) \cong \mathbb{N}$

② $K(*) = Gr(Vect^{iso}(*)) = Gr(\mathbb{N}) = \mathbb{Z}$

In fact, $X \simeq * \Rightarrow K(X) \cong \mathbb{Z}$



The Functor K

Defn: $K(X) = \{E - n\} / E - n = E' - m \Leftrightarrow \overset{n=m \text{ and}}{E \oplus m \cong E' \oplus n}$

Defn/Rmk. The assignment $X \mapsto K(X)$ defines a contravariant functor $\overset{k\mathrm{Top} := \text{Compact, Hausdorff Top}}{\cancel{\mathrm{Top}}} \to \mathrm{Ring}$

$$\begin{array}{ccc} X & \mapsto & K(X) \\ f \downarrow & \mapsto & \uparrow f^* \\ Y & \mapsto & K(Y) \end{array}$$

w/ $f^*(E) = f^*E$ pullback bundle

Prop. K factors through $\mathrm{Ho}k\mathrm{Top}$: $f \simeq g \Rightarrow f^* \cong g^*$

Reduced K-theory: For $x_0 \in X$, inclusion $x_0 \overset{i}{\hookrightarrow} X$ defines "dimension map" $K(X) \xrightarrow{i^*} K(*) \cong \mathbb{Z}$

Defn. $\tilde{K}(X) = \ker(K(X) \xrightarrow{i^*} K(x_0) \cong \mathbb{Z})$ 0-dim'l virtual bundles

$\Rightarrow$ $\tilde{K}(X)$ has ring structure and $\tilde{K} : k\mathrm{Top}_* \to \mathrm{Ring}$ functor

Rmk. $\tilde{K}(X) \cong \mathrm{Vect}^{iso}(X) /$ stable equivalence $E \sim_s E' \Leftrightarrow \exists\, n,m$ s.t. $E \oplus \varepsilon^n \cong E' \oplus \varepsilon^m$

Rmk. for X unbased, $K(X) = \tilde{K}(X_+)$ $\quad X \sqcup *$

for X based, $K(X) = \tilde{K}(X) \oplus \mathbb{Z}$

Prop. $K(X) = [X_+, BU \times \mathbb{Z}]_*$ and $\tilde{K}(X) = [X, BU \times \mathbb{Z}]_*$ lets us extend to non-compact spaces

Bott Periodicity… version 1

Central idea: Define an external product $\mu: K(X) \otimes K(Y) \to K(X \times Y)$
$x \otimes y \mapsto \pi_X^*(x) \cdot \pi_Y^*(y)$

Thm - There is an isomorphism $\mu: K(X) \otimes K(S^2) \xrightarrow{\cong} K(X \times S^2)$

Pf idea/
- H = canonical line bundle over $\mathbb{CP}^1 = S^2$
- Show $K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X) \otimes K(S^2) \xrightarrow{\mu} K(X \times S^2)$ is isom (long)
- Take $X = *$, then $\mathbb{Z}[H]/(H-1)^2 \xrightarrow{\cong} K(S^2)$

Reduced version: use LES for pair $(X \times Y, X \vee Y)$ to get external product
$\tilde{\mu}: \tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \wedge Y)$

Thm. If $Y = S^2$, then $\tilde{\mu}$ isom.

Thm (B.P. v1) - There is an isomorphism $\tilde{K}(X) \xrightarrow{\otimes (H-1)} \tilde{K}(X) \otimes \tilde{K}(S^2) \xrightarrow{\tilde{\mu}} \tilde{K}(X \wedge S^2) \cong \tilde{K}(\Sigma^2 X)$.

Pf idea/
- $\tilde{K}(S^2)$ is inf. cyclic w/ gen $(H-1)$ "Bott element"
- $\tilde{\mu}$ is isom
- $X \wedge S^2 \cong \Sigma^2 X$

Bott Periodicity ... Version 2

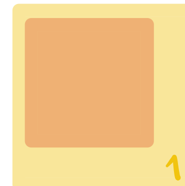
Prop: $\mathrm{Vect}^n_{\mathbb{C}}(X) \longleftrightarrow [X, BU(n)]$ for $BU(n) = Gr_n(\mathbb{C}^\infty)$ or classifying sp. of $U(n)$

Can include $U(n) \hookrightarrow U(n+1)$ and take colimit $U := \mathrm{colim}_n U(n)$.

Note: Since $U \overset{\swarrow \text{Exc}}{\simeq} \Omega BU$, can compute $\pi_i(BU) \cong \pi_{i-1}(\Omega BU) \cong \pi_{i-1}(U)$

Thm (V2) - The htpy gps of U are 2-periodic: $\pi_i(U) = \begin{cases} \mathbb{Z} & i \text{ odd} \\ 0 & i \text{ even} \end{cases}$

or there is a weak equiv $\Omega^2 BU \xrightarrow{\simeq} BU \times \mathbb{Z}$.



$U(n) \hookrightarrow U(n+1)$

Pf idea / requires lots of background! Main idea:

- relate π_k of $U(n)$, $SU(n)$, and $Gr_n(\mathbb{C}^{2n})$ for $n >> k$
- use LES and Morse theory to show
  $$\pi_k(U) \cong \pi_{k+1}(BU) \xrightarrow{\cong} \pi_{k+1}(SU(2n)) \cong \pi_{k+2}(U(n))$$
- Pf gives map $BU \to \Omega SU$ realizing isom $\qquad \Omega U \simeq BU \times \mathbb{Z}$

V1 $\Leftrightarrow$ V2: $\tilde{K}(X) = [X, BU \times \mathbb{Z}]_*$ and $\tilde{K}(\Sigma^2 X) = [\Sigma^2 X, BU \times \mathbb{Z}]_*$
$$= [X, \Omega^2(BU \times \mathbb{Z})]_*$$
$$= [X, \Omega^2 BU]_*$$

By Yoneda, $BU \times \mathbb{Z} \simeq \Omega^2 BU \Leftrightarrow [X, BU \times \mathbb{Z}]_* \cong [X, \Omega^2 BU]_* \Leftrightarrow \tilde{K}(X) \cong \tilde{K}(\Sigma^2 X)$

K-THEORY AS A GENERALIZED COHOMOLOGY THEORY

Defn - A reduced gen'lized cohom. theory $\tilde{E}^*$ consists of contravar. functors $\tilde{E}^n: HoTop_* \to Ab$ satisfying the following axioms:

1. Exactness: if $A \rightarrowtail X$ cofibration, then $\tilde{E}^n(X/A) \to \tilde{E}^n(X) \to \tilde{E}^n(A)$ exact
2. Suspension: $\exists$ nat'l isoms $\tilde{E}^n(X) \cong \tilde{E}^{n+1}(\Sigma X)$
3. Additivity: if $X = \bigvee X_i$ then $X_i \hookrightarrow X$ induce isom $\tilde{E}^n(X) \xrightarrow{\cong} \prod \tilde{E}^n(X_i)$
4. Weak Equivalence: if $X \xrightarrow{f} Y$ weak equiv then $\tilde{E}^*(Y) \xrightarrow{f^*} \tilde{E}^*(X)$ isom

Define $\tilde{K}^n(X) = \begin{cases} \tilde{K}(\Sigma^n X) & n \le 0 \\ \tilde{K}^{n-2}(X) & n > 0 \end{cases} \overset{B.P.}{\cong} \begin{cases} \tilde{K}(X) & n \text{ even} \\ \tilde{K}(\Sigma X) & n \text{ odd} \end{cases}$

1. Hatcher Prop 2.9
2. n odd: $\tilde{K}^{n+1}(\Sigma X) \cong \tilde{K}(\Sigma X) \cong \tilde{K}^n(X)$
   n even: $\tilde{K}^{n+1}(\Sigma X) \cong \tilde{K}(\Sigma^2 X) \cong \tilde{K}(X) \cong \tilde{K}^n(X)$
3. follows from LES (p.53 of Hatcher)
4. follows from working w/ "nice" spaces

*5. Dimension: $\tilde{E}^n(S^0) = 0$ for $n \neq 0$

✗5. $\tilde{K}(S^0) \neq 0$ infinitely often

Thus $\tilde{K}^*$ is reduced gen'lized cohomology theory

Rmk. K extends to unreduced gen'lized cohomology theory

K-THEORY AS A GENERALIZED COHOMOLOGY THEORY (cont.)

Thm. Every gen'lized cohom. theory corresponds to an Ω-spectrum. $\{E_n\}_n$, $E_n \xrightarrow{\simeq} \Omega E_{n+1}$

Recall $K(X) = [X_+, BU \times \mathbb{Z}]_*$

The topological K-theory spectrum is $KU_n = \begin{cases} BU \times \mathbb{Z} & n \text{ even} \\ \Omega BU & n \text{ odd} \end{cases}$

This means $K^n(X) = [X_+, KU_n]_*$

Application: Compute $K^n(S^k)$

$$K^n(S^k) = [S^k_+, KU_n]_* = \pi_k(KU_n) = \begin{cases} \pi_k(BU \times \mathbb{Z}) & n \text{ even} \\ \pi_k(\Omega BU) & n \text{ odd} \end{cases}$$

$$\Rightarrow K^n(S^k) = \begin{cases} \mathbb{Z} & n \equiv k \pmod 2 \quad k \neq 0, \quad \mathbb{Z} \oplus \mathbb{Z} \quad k = 0 \\ 0 & n \not\equiv k \pmod 2 \end{cases}$$

FINAL REMARKS ...

SUMMARY:

- $K(X)$ is commutative ring formed from vector bundles over X
- Also have reduced $\tilde{K}(X)$ w/ $K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$. Both $K, \tilde{K}$ are functors.
- Bott Periodicity extends $K, \tilde{K}$ to gen'lized cohomology theory w/ corresponding Ω-spectrum $BU \times \mathbb{Z}$

OTHER DIRECTIONS:

- Real K-theory + real Bott Periodicity
- Algebraic K-theory
- Equivariant K-theory
- + more ...

Thanks for Listening!