Chromatic Homotopy Theory Seminar
(summer edition)

Bousfield Localization

of Spectra

Maxine Calle

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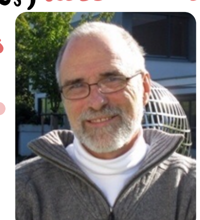
Overview

Localization: we can "invert" operators, collections of maps, etc...
the "local" objects are those which already see these maps as invertible
$\Rightarrow$ want to systematically make objs local

Bousfield Localization: axiomatizes this idea for spectra
for given E_*, get functor $L_E : \mathcal{S} \to \mathcal{S}$
which localizes wrt "E_*-equivalences"

Motivation: want to understand $[X,Y]_*$ given info about E_*X, E_*Y (e.g. in SS's)
BUT can't get info E_* doesn't "see"
$\Rightarrow$ localization s.t. htpy info determined by E_*-homological info

History: Serre (1950s) break up htpy/homology gps
Quillen, Sullivan, Bousfield-Kan (1960s-70s) associate spaces
Morava, Ravenel (1980s) used techniques



Ideas of "Localization" in homotopy theory

of rings: $S \subseteq R$ multiplicative subset, "systematically add in multiplicative inverses" of $s \in S$

better: invert $\mu_s : R \to R$ for $s \in S$

in categories: systematically "invert" a suitable collection of maps $S \subseteq \mathcal{C}$

↳ an obj $X \in \mathcal{C}$ is S-local: for all $A \xrightarrow{f} B$ in S, $\mathcal{C}(B,X) \xrightarrow{\sim} \mathcal{C}(A,X)$ is w.e.

↳ a map $A \to B \in \mathcal{C}$ is S-equivalence: for S-local X,

$$\mathcal{C}(B,X) \xrightarrow{\sim} \mathcal{C}(A,X) \text{ w.e.}$$

↳ the S-localization of $X \in \mathcal{C}$: S-equiv $X \to Y$ for Y S-local

Examples (in spaces): $S = \{f : \emptyset \hookrightarrow Y \text{ (for } Y \neq \emptyset) : Y \in \mathrm{Top}\}$

S-local X = $\mathrm{Top}(Y,X) \xrightarrow{\sim} \mathrm{Top}(\emptyset, X)$ so X is weakly contractible

S-equivalences $A \to B$ = $\mathrm{Top}(B,X) \xrightarrow{\sim} \mathrm{Top}(A,X)$ for all w.c. X

S-localization of X = $X \to *$

$S = \{S^n \to *\}$ singleton

S-local X = $\Omega^n X \simeq *$ for any basept

S-equiv = $(n-1)$-cntd maps

S-localization = $X \to Y$ $\quad \pi_k(X) \xrightarrow{\cong} \pi_k(Y)$ $\quad 0 \leq k < n$

and $\pi_k(Y) = 0$ $\quad k \geq n$.

In a stable setting

Stabilization as localization: "invert suspension functor Σ" to get to stable homotopy category $\mathcal{S}$ w/ functor $\Sigma^\infty : Ho(Top_*) \to \mathcal{S} \xleftarrow{\simeq} Ho(Sp)$

rationalization: $S = \{\text{multiply-by-}m\text{ maps } S^n \to S^n \mid n \in \mathbb{Z}, m > 0\}$

S-local spectra $= \pi_* Y \xrightarrow[m]{\cong} \pi_* Y \quad \forall m > 0$

$\iff \pi_* Y \xrightarrow{\cong} \pi_* Y \otimes \mathbb{Q}$ "rational spectra"

S-equiv $A \to B$ $= \pi_* A \otimes \mathbb{Q} \xrightarrow{\cong} \pi_* B \otimes \mathbb{Q}$

$\iff H_*(A, \mathbb{Q}) \xrightarrow{\cong} H_*(B, \mathbb{Q})$ "rational equivalences"

S-localization of X $= X \mapsto H\mathbb{Q} \wedge X =: X_{\mathbb{Q}}$ "rationalization"

$= M\mathbb{Q}$ Moore spectrum

p-inversion: $S = \{\text{multiply-by-}p\text{ maps } S^n \to S^n : n \in \mathbb{Z}\}$

S-local spectra $= \pi_* Y$ are $\mathbb{Z}[1/p]$-modules

S-equivs $A \to B$ $= \pi_* A[1/p] \xrightarrow{\cong} \pi_* B[1/p]$

S-localization of X $= X \mapsto \mathbb{S}[1/p] \wedge X$ for $\mathbb{S}[1/p] =$ Moore spectrum of $\mathbb{Z}[1/p]$

$= \operatorname{hocolim}(\mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \dots)$

p-localization: $S = \{\text{mult. by } m \text{ for } (m,p) = 1 \; S^n \to S^n\}$

replace $\mathbb{Z}[1/p]$ w/ $\mathbb{Z}_{(p)}$

So localization is $X \mapsto X \wedge M(\mathbb{Z}_{(p)})$

Bousfield Localization of Spectra

<u>Idea</u>: for a given spectrum E, $S =$ "E_*-equivalences"

<u>Notation</u>: E_* is homology theory $E_n X = \pi_n(E \wedge X) = [\Sigma^n \mathbb{S}, E \wedge X]$
$[X, Y]_* =$ graded Ab gp w/ $[X,Y]_n = [\Sigma^n X, Y]$

<u>Defns</u>: X is E_*-acyclic: $E_* X = 0$ i.e. $E \wedge X \simeq *$

X is E_*-local: $[Y, X]_* = 0$ for Y E_*-acyclic

$X \xrightarrow{f} Y$ is E_*-equivalence: $E \wedge f : E \wedge X \xrightarrow{\sim} E \wedge Y$

<u>Lem</u>: X is E_*-local iff each E_*-equiv $A \xrightarrow{f} B$ induces isom $[A, X]_* \xleftarrow{\cong} [B, X]_*$.

Pf/ ($\Leftarrow$) Shorter: if Y is E_*-acyclic, then $* \to Y$ is E_*-equiv
($\Rightarrow$) Longer: E_*-equiv $A \to B$ has E_*-acyclic cofiber

<u>Ex</u>: module spectrum M over ring spectrum E is E_*-local
if Y E_*-acyclic, then $Y \xrightarrow{f} M$ factors

$$\begin{tikzcd} & * \arrow[d, phantom, "\text{is}"] & & \\ Y \arrow[r, hook] \arrow[rrr, bend right, "f"'] & E \wedge Y \arrow[r, "1 \wedge f"] & E \wedge M \arrow[r, "\mu"] & M \end{tikzcd}$$

$\Rightarrow [Y, M]_* = 0$

e.g. $M = E \wedge X$ for any spectrum X

Localization

$X \xrightarrow{f} Y$ is E_*-localization of X: Y is E_*-local and f is E_*-equivalence

Thm/Defn: Every E_* has E_*-localization functor $L_E : \mathcal{S} \to \mathcal{S}$ w/ nat'l equiv $1_{\mathcal{S}} \xRightarrow{\eta_E} L_E$ s.t.

(i) η_X is E_*-equiv for all X

(ii) for any E_*-equiv $X \xrightarrow{f} Y$, then TFDC:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & L_E X \arrow[dl, dotted, "\exists !"] \\ Y & \end{tikzcd}$$

Prop: (1) L_E is unique up to htpy

(2) $L_E^2 = L_E$

(3) takes cofiber seq to cofiber seqs

Existence: roughly, $L_E X \simeq \underset{\substack{X \to Y \\ E_*\text{-equiv}}}{\operatorname{hocolim}} Y$ ← set theory issues

Bousfield-Smith cardinality argument:

1. there is spectrum A s.t.
   - (i) X is E_*-local $\iff [A, X]_* = 0$
   - (ii) A is E_*-acyclic
   - (iii) A is "κ-small" for inf. cardinal κ
2. use small object argument + homotopy cofiber stuff

Connect to Model Categories on spectra

- cofibr = usual
- w.e. = E_*-equiv
- fibr = lifting property

$\Rightarrow$ fibrant objs = E_*-local Ω-spectra

localization = fibrant replacement

Examples

<u>Defn</u>- L_E is smashing: $L_E X \simeq X \wedge L_E \mathbb{S}$

1. rationalization: $E = H\mathbb{Q} = M\mathbb{Q}$, $L_E X \simeq X \wedge H\mathbb{Q} = X \wedge E$
2. p-inversion: $E = M(\mathbb{Z}[1/p])$, $L_E X \simeq X \wedge M(\mathbb{Z}[1/p])$
3. p-localization: $E = M(\mathbb{Z}_{(p)})$, $L_E X \simeq X \wedge M(\mathbb{Z}_{(p)})$

Non-smashing examples:

4. $E = \mathbb{S}$: homology $= \pi_*$

   $L_E X$ = Ω-spectrum equiv to X
5. p-completion $E = M(\mathbb{Z}/p)$: $L_E X \simeq X^\wedge_p$

$$X^\wedge_p = \operatorname{holim}\{\cdots \to X \wedge M(\mathbb{Z}/p^2) \to X \wedge M(\mathbb{Z}/p)\}$$

Bonus info

Thm. Localization of E_∞-ring spectra are E_∞.

↳ in EKMM: email from Hopkins + McClure

↳ in Ravenel: E_*-localization of $\mathbb{S}$ is comm. ring spectrum

"Local-to-Global" Let E, F, and X be spectra s.t. $E_*(L_F X) = 0$. Then there is htpy pullback

$$\begin{tikzcd} L_{E \vee F} X \arrow[r, "\exists!"] \arrow[d, "\exists!"'] \arrow[dr, phantom, "\lrcorner"] & L_E X \arrow[d] \\ L_F X \arrow[r, "L_F(\eta_E)"'] & L_F L_E X \end{tikzcd}$$

Sullivan-Arithmetic Square: $E = \bigvee_p M(\mathbb{Z}/p)$, $F = H\mathbb{Q} = M\mathbb{Q}$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_p L_p X \arrow[d] \\ L_{\mathbb{Q}} X \arrow[r] & L_{\mathbb{Q}}(\prod_p L_p X) \end{tikzcd}$$

References

- Lurie Lecture 20
- Péroux master project :)
- Ravenel Localization wrt certain periodic homology theories
- Bauer Chapter 6 in TMF
- EKMM Chapter 8
- Lawson Intro to Bousfield localization

THANKS for LISTENING!