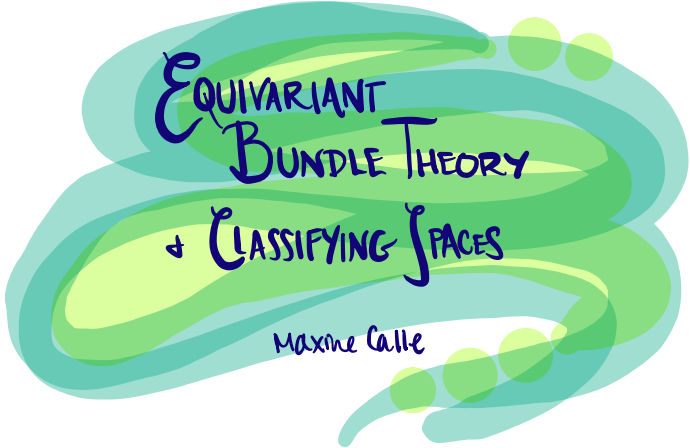


eCHT equivariant reading seminar
Fall 2021



EQUIVARIANT
BUNDLE THEORY
& CLASSIFYING SPACES

Maxime Calle

Motivating Question

What is the "right" way to generalize non-equivariant bundle theory to the equivariant setting?

Prelude: The non-equivariant story

§1 - Definitions: • The easier case $P = G \times \Pi$
• The harder case $P = G \times \Pi$ + the hardest case

§2 - Universal (Π, P) -bundles: • Definition
• Models

§3 - Fixed point thms

Overview

Ch. VIII of
May's *Algebraic Topology*

• Exercises at the end •

PRELUDE: The non-equivariant Setting

Defn - A principal Π -bundle over B consists of a Π -space P and Π -map $P \xrightarrow{p} B$ (w/ $B \xrightarrow{2\pi}$ trivially) which is locally trivial

$$\begin{array}{ccc} \hookrightarrow \exists \text{ cover } \{U_i\} \text{ of } B \text{ w/ } \Pi\text{-homeomorphism } \phi_i: P^{-1}U_i & \xrightarrow{\cong} & U_i \times \Pi \\ \downarrow p & & \downarrow \text{pr} \\ & & U_i \end{array}$$

Note: Triviality condition implies $P \xrightarrow{2\pi}$ freely and P factors through $P/\Pi \xrightarrow{\cong} B$ so B "is" the orbit space $P \xrightarrow{2\pi}$

Defn - Let F be a Π -Space. A fiber bundle w/ fiber F and structure gp Π consists of $E \xrightarrow{p} B$ w/ local triv. $\phi_i: P^{-1}U_i \cong U_i \times F$ s.t.
 $\phi_i \phi_j^{-1}(u, f) = (u, g_{ij}(u)(f))$ for $g_{ij}: U_i \cap U_j \rightarrow \Pi$ (w/ $\Pi \subseteq \text{Aut}(F)$)

Thm - Let Π cpr Lie and $F \xrightarrow{2\pi}, B$ spaces. Then

$$\left\{ \begin{array}{l} \text{fiber bundles over } B \\ \text{w/ fibr } F \text{ + str. gp } \Pi \end{array} \right\} \begin{array}{c} \rightarrow \\ \leftarrow \end{array} \left\{ \begin{array}{l} \text{principal} \\ \Pi\text{-bundles} \\ \text{over } B \end{array} \right\}$$

"pf" \leftarrow , use $F \rightarrow *$ to induce $E \times_{\Pi} F \rightarrow E \times_{\Pi} * = B$
 \rightarrow , define $\mathcal{P} \subseteq \text{Map}(F, E)$ "admissible"

Examples (1) If Π discrete, then a principal Π -bundle (w/ connected total space) is a covering space whose deck transformation gp is Π .

(2) $\Pi = \text{GL}_n(\mathbb{R})$ and $F = \mathbb{R}^n \Rightarrow$ real vector bundles of rk n .

Universal Principal Π -bundle - Every principal Π -bundle arises as pullback of a universal one

$$\begin{array}{ccc} P & \rightarrow & E\Pi \\ \downarrow & \Gamma & \downarrow \\ B & \rightarrow & B\Pi \end{array} \quad \left\{ \begin{array}{l} \text{diff. model for } B\Pi \\ \text{e.g. } \Pi \text{ discrete, } B\Pi = K(\Pi, 1) \text{ and } E\Pi = \text{univ. cover} \end{array} \right.$$

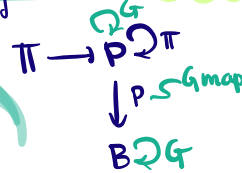
The Equivariant Setting

§1.

Idea: Introduce equivariance gp G which acts on principal π -bundle $P \xrightarrow{\pi} B$

Easier Case: G acts trivially on π ($P = G \times \pi$)
 $\Rightarrow P \curvearrowright G$ comm. w/ $P \curvearrowright \pi \Rightarrow P \curvearrowright G \times \pi$

Pictorially:



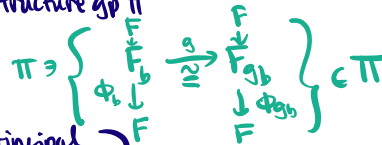
Defn - A principal (G, π) -bundle consists of a G -space P w/ free action $P \curvearrowright \pi$ and G -map $p: P \rightarrow P/\pi =: B$.

Defn - A G -bundle w/ fiber F and structural gp π is a G -map $E \xrightarrow{p} B$ s.t.

- (i) is it a fiber bundle w/ fiber F + str. gp π non-equivariantly
- (ii) " G acts by bundle morphisms w/ structure gp π "

Thm - Let G, π opt Lie and F, B be spaces. Then

$$\left\{ \begin{array}{l} G\text{-bundles over } B \\ \text{w/ fiber } F \text{ and} \\ \text{structure gp } \pi \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{principal} \\ (G, \pi)\text{-bundles} \\ \text{over } B \end{array} \right\}$$



Examples (1) G -equivariant vector bundles: $\text{rank } n \Rightarrow \left\{ \begin{array}{l} \pi = O(n) \\ F = \mathbb{R}^n \end{array} \right.$

e.g. $TM \rightarrow M$ G -mfld M

(2) Let $p: E \rightarrow B$ be a covering G -map w/ finite fibers, $|p^{-1}(b)| = n$. Then:

$$\left\{ \begin{array}{l} \pi = \Sigma_n \\ F = \{1, \dots, n\} = \underline{n} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} G\text{-bundle w/ fiber} \\ \underline{n} \text{ + str. gp } \Sigma_n \end{array} \right. \quad \underline{n} \rightarrow E \cong P \times_{\Sigma_n} \underline{n}$$

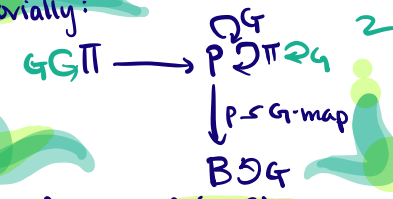
$P = \{ \underline{n} \rightarrow E \text{ which are bij onto fibers of } p \}$



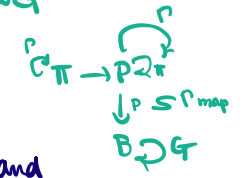
More general (Harder) Cases ($\Gamma = \Pi \times G$ or $P = ?$)

What if G acts on Π nontrivially?

Pictorially:



\curvearrowright actions $P \curvearrowright G$ and $P \curvearrowright \Pi$ don't commute but "twisted" by $\Pi \curvearrowright G \Rightarrow P \curvearrowright \Pi \times G$
 more generally: $1 \rightarrow \Pi \rightarrow \Gamma \rightarrow G \rightarrow 1$



Defn - A principal (Π, Γ) -bundle consists of a Γ -space P w/ free action $P \curvearrowright \Pi$ and "projection to orbits" Γ -map $p: P \rightarrow P/\Pi = B$.

Rmk Can define Γ -bundle w/ fiber F and str. gp Π like before (but a bit more complicated)
 Same stuff works:

Other Stuff

- What are trivial (Π, Γ) -bundles? Let $K \leq G$ and $\mathcal{L} \leq \Gamma$, $q: \Gamma \rightarrow G$
 If (i) $\mathcal{L} \cap \Pi = e$, (ii) $q: \mathcal{L} \xrightarrow{\cong} K$, then $K \cong \text{space } U$
 $\curvearrowright G/P/\mathcal{L} \curvearrowright \Pi$ freely $q \times \text{id}: P \times \mathcal{L} U \rightarrow G \times K U$

Interesting Examples
 • algebraic + topological K-theory

- for U \mathcal{L} -space by pulling back along q . Bundle is trivial if equiv. to one of these.
- If P (and so B) is completely regular, then $P \xrightarrow{p} B$ is locally trivial.
 \curvearrowright e.g. mflds, CW cpx, top. gps.
- Pullbacks of numerable (Π, Γ) -bundles along htpe maps are equivalent

CLASSIFICATION

§2.

Recall: \mathcal{F} is a family of subgps closed under conjugacy + $E\mathcal{F}$ is "universal \mathcal{F} -space"

? + subgps

$$E\mathcal{F} = \mathcal{F} \times_{\mathcal{F}} U \quad \text{for } \mathcal{F}: \text{Fun}(\mathcal{G}, U) \rightarrow \text{GM} \text{ and } \mathcal{F}: \mathcal{H}\mathcal{G} \rightarrow \text{Set}$$

← Elmendorf
 $\mathcal{G}/H \mapsto \begin{cases} * & H \in \mathcal{F} \\ \emptyset & H \notin \mathcal{F} \end{cases}$

- $E\mathcal{F} = E(\mathcal{F}|_H)$ as H -spaces, for $\mathcal{F}|_H = \{K \in \mathcal{F} \mid K \leq H\}$
- For $H \in \mathcal{F}$, $E(\mathcal{F}|_H)^H = E(\mathcal{F}|_H^H)$ as $W_G H$ -spaces, for $\mathcal{F}|_H^H = \{K \in \mathcal{F} \mid K \leq H \text{ s.t. } H \leq K \leq N_G H\}$

Defn (Universal (π, P) -bundle)

- Let $\mathcal{F}(\pi, P) = \{\lambda \in P : \lambda \cap \pi = e\}$
 $\hookrightarrow \mathcal{F}$ -spaces are π -free P -spaces
- Write $E(\pi, P) = E\mathcal{F}(\pi, P)$ and $B(\pi, P) = E\mathcal{F}(\pi, P)/\pi$, so have principal (π, P) -bundle $E(\pi, P) \xrightarrow{\pi} B(\pi, P)$
 \hookrightarrow universal π -free P -space \hookrightarrow orbit space

Note: $B(\pi, P)$ models BTT as particular G -space

Thm - $E(\pi, P) \rightarrow B(\pi, P)$ is universal, i.e. there is a bijection

$$\left\{ \begin{array}{l} \text{equiv. classes} \\ \text{of principal} \\ (\pi, P)\text{-bundles} \\ \text{over } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} h\text{-homotopy} \\ \text{classes of} \\ G\text{-maps} \\ X \rightarrow B(\pi, P) \end{array} \right\} \quad \text{i.e.} \quad \begin{array}{ccc} P & \rightarrow & E(\pi, P) \\ \downarrow & & \downarrow \\ X & \rightarrow & B(\pi, P) \end{array}$$

Notation: If $P = G \times \pi$, write $E_G \pi := E\mathcal{F}(\pi, P)$ and $B_G \pi := E\mathcal{F}(\pi, P)/\pi$.

EXAMPLES OF CLASSIFYING SPACES

§2.

Goal: Models for $E(\pi, \Gamma)$ (and therefore models for $B(\pi, \Gamma)$)

Claim/Thm - A principal (π, Γ) -bundle E is univ $\Leftrightarrow E^\Lambda \simeq *$ for all $\Lambda \in \Gamma$ s.t. $\Lambda \cap \pi = e$

① Let $\text{Sec}(E_G, E_\Gamma) \subseteq \text{Map}(E_G, E_\Gamma)$ in Γ -spaces (where $E_G \simeq \mathbb{Z}^\Gamma$ by $q: \Gamma \rightarrow G$) be Γ -sections of $E_q: E_\Gamma \rightarrow E_G$.

Thm. This is model for univ. principal (π, Γ) -bundle \Rightarrow so $\text{Sec}(E_G, E_\Gamma)/\pi$ is a model for $B(\pi, \Gamma)$

② Other model: For G discrete, π discrete or cpt Lie, $\Gamma = \pi \times G$

$\rightarrow \text{Ob} = G \quad \text{Mor} = \begin{matrix} \circ & \xrightarrow{g} & \circ' \end{matrix}$

Thm. Guillou-May-Merling (2015): $\text{BCat}(\tilde{G}, \tilde{\pi}) \rightarrow \text{BCat}(G, \pi)$ is universal

Connecting $B_G \pi$ and $B\pi$

Simplify to $\Gamma = G \times \pi$: In this case, $\text{Sec}(E_G, E_\Gamma) = \text{Map}(E_G, E_\pi)$

so $\text{Map}(E_G, E_\pi)/\pi$ models $B_G \pi$

Let $\mathcal{B}_G \pi(X) = [X, B_G \pi]_G$ and $\mathcal{B}\pi(X) = [X, B\pi]$

Note: By adjunction, a G -map $X \rightarrow \text{Map}(E_G/B\pi) \leftrightarrow E_G \times_G X \rightarrow B\pi$

so if have $B_G \pi \xrightarrow{\alpha} \text{Map}(E_G, B\pi)$, then get $\mathcal{B}_G \pi(X) \rightarrow \mathcal{B}\pi(E_G \times_G X)$

" $\text{Map}(E_G, E_\pi)/\pi \xrightarrow{\alpha} \text{induced by } E(G \times \pi) \rightarrow B(G \times \pi)$

Rmk. More generally, $E_\Gamma \rightarrow E_\Gamma$ induces $\alpha: B(\pi, \Gamma) \rightarrow \text{Sec}(E_G, E_\Gamma) = \left\{ \begin{matrix} E_G \xrightarrow{\alpha} B\Gamma \\ \downarrow B\alpha \\ B_G \end{matrix} \right\}$

Q. How much info does $\mathcal{B}_G \pi(X) \rightarrow \mathcal{B}\pi(E_G \times_G X)$ lose?

Thm(s) - If π discrete, α is homeom. (if Γ discrete, α homeom.)
If G cpt Lie and π Abelian cpt Lie, then d.w.e.

FIXED POINT THMS

§3.

Notation: Let $K \leq G$, $\Lambda \leq \Gamma$ s.t. $\Lambda \cap \Pi = e$ and $q: \Gamma \rightarrow G$ maps $q(\Lambda) \cong K$.
Set $\Pi^\Lambda = \Pi \cap N_\Gamma \Lambda = \Pi \cap Z_\Gamma \Lambda$

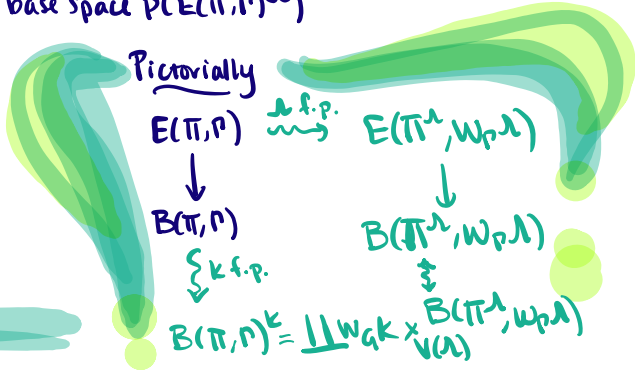
Rmk: $E(\Pi, \Gamma)^\Lambda = E(\mathfrak{f}^\Lambda)$ for $\mathfrak{f}^\Lambda = \{ \Lambda' / \Lambda \mid \Lambda \leq \Lambda' \leq N_\Gamma \Lambda \}$ from earlier ch
is univ. principal $(\Pi^\Lambda, W_\Gamma \Lambda)$ -bundle w/ base space $P(E(\Pi, \Gamma)^\Lambda)$

Thm - For $K \leq G$, $B(\Pi, \Gamma)^K = \coprod_{\substack{\Pi\text{-conj. classes} \\ \text{of } \Lambda \text{ s.t. } \dots}} B(\Pi^\Lambda)$

As a $W_G K$ -space;

$$= \coprod_{\substack{q^{-1}(N_G K)\text{-conj.} \\ \text{classes of } \Lambda \text{ s.t. } \dots}} W_G K \times_{V(\Lambda)} B(\Pi^\Lambda, W_\Gamma \Lambda)$$

$\swarrow W_\Gamma \Lambda / \Pi^\Lambda$



Simplify to $\Gamma = G \times \Pi$

Λ is of the form $\Lambda(p) = \{ (h, \rho(h)) \mid h \in H \leq G, \rho: H \rightarrow \Pi \}$
So can re-express thm in terms of $\rho: H \rightarrow \Pi$

Thm (again) - $B^H = \coprod_{[\rho] \in \text{Rep}(H, \Pi)} B(\Pi^{\Lambda(p)})$ $\leftarrow G/H \times \Pi / \Pi\text{-conjugacy}$

In particular, $\Pi \rightarrow E^H \rightarrow P(E^H)$ is a principal Π -bundle

Rmk In general, $W_{G \times \Pi} \Lambda \neq V(\Lambda) \times \Pi^\Lambda$ so need general $1 \rightarrow \Pi \rightarrow \Gamma \rightarrow G \rightarrow 1$

Skipping: justification of $\left\{ \begin{array}{l} \text{Thm } H_G^*(B(\Pi, \Gamma)) \cong H^*(B\Gamma) \text{ w/ any coeff} \\ \text{Borel } H_G^*(B(\Pi, G \times \Pi)) \cong H^*(BG) \otimes H^*(B\Pi) \text{ as } H^*(BG)\text{-module} \\ \text{w/ field coeff} \end{array} \right.$

Q. What can we say about $\alpha^H: B(\Pi, \Gamma)^H \rightarrow \text{Sec}(EG, B\Gamma)^H$?

For $\Gamma = G \times \Pi$

$$B(\Pi, \Gamma)^H = \coprod_{[\rho] \in \text{Rep}(H, \Pi)} B(\Pi^{\Lambda(p)})$$

$\downarrow \alpha^H$

$$\text{Sec}(EG, B\Gamma)^H = \text{Map}(EG, BG \times B\Pi)^H = \text{Map}(E^H, B\Pi)$$

\rightsquigarrow restrict α^H to $B(\Pi^{\Lambda(p)})$:
adjoin to $B\gamma: BH \times B(\Pi^{\Lambda(p)}) \rightarrow B\Pi$
where $\gamma: H \times \Pi^{\Lambda(p)} \rightarrow \Pi$
 $(h, \pi) \rightarrow \rho(h)\pi$

THANKS FOR LISTENING 😊

Suggested Exercises

Just do the ones that look interesting to you!

From May:

- (1) Let $E \xrightarrow{p} B$ be a principal (Π, Γ) -bundle and $H \leq G$. Show B^H is disjoint union of $p(E^\lambda)$ for λ runs over Π -conjugacy classes of subgs $\lambda \leq \Gamma$ s.t. $\lambda \cap \Pi = e$ and $q: \Gamma \rightarrow G$ maps $q(\lambda) \cong H$.
- (2) Work out example 3.1 in the complex case

Concrete Exercises:

- (3) Stewart: For n odd, have $SO(n) \times \{\pm 1\} \cong O(n)$. What are principal $(SO(n), O(n))$ -bundles? How can we understand $E(SO(n), O(n)) \rightarrow B(SO(n), O(n))$ explicitly?
- (4) Find an example $1 \rightarrow \Pi \rightarrow \Gamma \rightarrow G \rightarrow 1$ w/ G -space B s.t. projection $B \times \Pi \rightarrow \Pi$ is not a trivial (Π, Γ) -bundle.
- (5) Let $G = C_2$ and $\Pi = O(2)$, $\Gamma = G \times \Pi$. What is $B_{C_2} O(2)$? What is $B_{C_2} O(2)^{C_2}$?

Open-ended questions: i.e. idk the answer or if these are good/interesting questions

- (6) In the non-equivariant setting, a principal Π -bundle $E \xrightarrow{p} B$ is trivial iff it admits a section $s: B \rightarrow E$. This implies, e.g., that $E \times_B E \rightarrow E$ is a principal Π -bundle.
 - (a) Can you come up with a (partial) generalization of this to the equivariant setting?
 - (b) Or, characterize when the pullback $E \times_B E \rightarrow E$ is a principal (Π, Γ) -bundle?
- (7) Some other non-equivariant stuff in equivariant setting:
 - (a) Let $\Pi \supseteq \Gamma$ and $\Pi' \leq \Pi$. When is $\Pi \rightarrow \Pi/\Pi'$ a principal $(\Pi, \Pi \rtimes G)$ -bundle?
 - (b) Let $E \rightarrow B$ be a principal (Π, Γ) -bundle and $\Pi' \leq \Pi$. Under what conditions is $E \rightarrow E/\Pi'$ a principal (Π', Γ) -bundle?