eCHT equivariant reading seminar
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# Equivariant Bundle Theory & Classifying Spaces

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**Motivating Question**

What is the "right" way to generalize non-equivariant bundle theory to the equivariant setting?

**Overview**

Ch VII of May's Alaska Notes

Prelude: The non-equivariant story

§1 - Definitions:
- The easier case $\Gamma = G \times \Pi$
- The harder case $\Gamma = G \rtimes \Pi$ + the hardest case

§2 - Universal $(\Pi, \Gamma)$-bundles:
- Definition
- Models

§3 - Fixed point thms

Exercises at the end

# PRELUDE: The non-equivariant Setting

**Defn** - A principal $\Pi$-bundle over $B$ consists of a $\Pi$-space $P$ and $\Pi$-map $P \xrightarrow{p} B$ ($w/\ B \circlearrowleft \Pi$ trivially) which is locally trivial

$\rightsquigarrow$ $\exists$ cover $\{U\}$ of $B$ w/ $\Pi$-homeomorphism $\Phi_U : p^{-1}U \xrightarrow{\cong} U \times \Pi \circlearrowleft \Pi$

$$\begin{tikzcd} p^{-1}U \arrow[r, "\cong"] \arrow[dr, "p"'] & U \times \Pi \arrow[d] \\ & U \end{tikzcd}$$

**Note**: Triviality condition implies $P \circlearrowleft \Pi$ freely and $p$ factors through $P/\Pi \xrightarrow{\cong} B$ so $B$ "is" the orbit space $P \circlearrowleft \Pi$

**Defn** - Let $F$ be a $\Pi$-Space. A fiber bundle w/ fiber $F$ and structure gp $\Pi$ consists of $E \xrightarrow{p} B$ w/ local triv. $\phi_i : p^{-1}U_i \cong U_i \times F$ s.t. $\curvearrowleft F \circlearrowleft \Pi$ effectively

$$\phi_i \phi_j^{-1}(u, f) = (u, g_{ij}(u)(f)) \quad \text{for } g_{ij} : U_i \cap U_j \to \Pi \ (w/\ \Pi \subseteq \mathrm{Aut}(F))$$

**Thm** - Let $\Pi$ cpt Lie and $F \circlearrowleft \Pi$, $B$ spaces. Then

$$\left\{ \begin{array}{c} \text{fiber bundles over } B \\ \text{w/ fiber } F + \text{str. gp } \Pi \end{array} \right\} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \left\{ \begin{array}{c} \text{principal} \\ \Pi\text{-bundles} \\ \text{over } B \end{array} \right\}$$

"Pf" $\leftarrow$, use $F \to *$ to induce $E \times_\Pi F \to E \times_\Pi * = B$

$\rightarrow$, define $P \subseteq \mathrm{Map}(F, E)$ "admissible"

**Examples** (1) If $\Pi$ discrete, then a principal $\Pi$-bunde (w/ connected total space) is a covering space whose deck transformation gp is $\Pi$.

(2) $\Pi = GL_n(\mathbb{R})$ and $F = \mathbb{R}^n \Rightarrow$ real vector bundles of rk $n$.

**Universal Principal $\Pi$-bundle** - Every principal $\Pi$-bundle arises as pullback of a universal one

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & E\Pi \arrow[d] \\ B \arrow[r] & B\Pi \end{tikzcd}$$

$\exists$ diff. model for $B\Pi$
e.g. $\Pi$ discrete, $B\Pi = K(\Pi, 1)$ and $E\Pi$ = univ. cover

# The Equivariant Setting

## §1.

**Idea:** Introduce equivariance gp $G$ which acts on principal $\Pi$-bundle $\overset{\circlearrowleft \Pi}{P} \xrightarrow{p} B$

**Easier Case:** $G$ acts trivially on $\Pi$ ($P = G \times \Pi$)
$\rightsquigarrow$ $P \circlearrowleft G$ comm. w/ $P \circlearrowleft \Pi \Rightarrow P \circlearrowleft G \times \Pi$

Pictorially:

$$\begin{tikzcd} \Pi \arrow[r] & \overset{\circlearrowleft G}{P} \circlearrowleft \Pi \arrow[d, "p"] \\ & B \circlearrowleft G \end{tikzcd}$$

$p$ G-map

**Defn** - A principal $(G,\Pi)$-bundle consists of a $G$-space $P$ w/ free action $P \circlearrowleft \Pi$ and $G$-map $p: P \to P/\Pi =: B$.

**Defn** - A $G$-bundle w/ fiber $F$ and structural gp $\Pi$ is a $G$-map $E \xrightarrow{p} B$ s.t.
(i) is it a fiber bundle w/ fiber $F$ + str. gp $\Pi$ non-equivariantly
(ii) "$G$ acts by bundle morphisms w/ structure gp $\Pi$"

$$\Pi \ni \left\{ \begin{tikzcd} F \arrow[d] & F \arrow[d] \\ F_b \arrow[r, "g", "\cong"'] \arrow[d, "\Phi_b"] & F_{gb} \arrow[d, "\Phi_{gb}"] \\ F & F \end{tikzcd} \right\} \in \Pi$$

**Thm** - Let $G, \Pi$ cpt Lie and $F^{\circlearrowleft \Pi}$, $B$ be spaces. Then $\exists$

$$\left\{ \begin{matrix} G\text{-bundles over } B \\ \text{w/ fiber } F \text{ and} \\ \text{structure gp } \Pi \end{matrix} \right\} \rightleftarrows \left\{ \begin{matrix} \text{principal} \\ (G,\Pi)\text{-bundles} \\ \text{over } B \end{matrix} \right\}$$

**Examples** (1) $G$-equivariant vector bundles: + real + rk $n$ $\Rightarrow \begin{cases} \Pi = O(n) \\ F = \mathbb{R}^n \end{cases}$

e.g. $TM \to M$ $G$-mfld $M$

(2) Let $p: E \to B$ be a covering $G$-map w/ finite fibers, $|p^{-1}(b)| = n$. Then:

$\begin{cases} \Pi = \Sigma_n \\ F = \{1, \dots, n\} = \underline{n} \end{cases} \Rightarrow$ $G$-bundle w/ fiber $\underline{n}$ + str. gp $\Sigma_n$

$$\begin{tikzcd} \underline{n} \arrow[r] & E \cong P \times_{\Sigma_n} \underline{n} \arrow[d, "p"] \\ & B \end{tikzcd}$$

$P = \{ \underline{n} \to E$ which are bij onto fibers of $p \}$

## More general (Harder) Cases ($\Gamma = \Pi \rtimes G$ or $\Gamma = ?$)

§1.

What if $G$ acts on $\Pi$ nontrivially?

Pictorially:

$$\begin{array}{ccc} G \circlearrowright \Pi & \longrightarrow & G \circlearrowright P \circlearrowleft \Pi \circlearrowleft G \\ & & \big\downarrow p \text{ a } G\text{-map} \\ & & B \circlearrowleft G \end{array}$$

⇝ actions $P \circlearrowleft G$ and $P \circlearrowleft \Pi$ don't commute but "twisted" by $\Pi \circlearrowleft G$ $\Rightarrow P \circlearrowleft \Pi \rtimes G$

more generally: $1 \to \Pi \to \Gamma \to G \to 1$

$$\begin{array}{ccc} \Gamma \circlearrowright \Pi & \longrightarrow & \Gamma \circlearrowright P \circlearrowleft \Pi \\ & & \big\downarrow p \text{ a } \Gamma\text{ map} \\ & & B \circlearrowleft G \end{array}$$

**Defn** - A principal $(\Pi,\Gamma)$-bundle consists of a $\Gamma$-space $P$ w/ free action $P \circlearrowleft \Pi$ and "projection to orbits" $\Gamma$-map $p: P \longrightarrow P/\Pi =: B$.

**Rmk** Can define $\Gamma$-bundle w/ fiber $F$ and str. gp $\Pi$ like before (but a bit more complicated)

Same stuff works:

### Other Stuff

- What are trivial $(\Pi,\Gamma)$-bundles? Let $K \le G$ and $\Lambda \le \Gamma$, $q: \Gamma \to G$

If (i) $\Lambda \cap \Pi = e$, (ii) $q: \Lambda \xrightarrow{\cong} K$, then $K$-space $U$

?: $\Gamma/\Lambda \circlearrowleft \Pi$ freely

$$q \times id: \Gamma \times_\Lambda U \longrightarrow G \times_K U$$

for $U$ $\Lambda$-space by pulling back along $q$. Bundle is trivial if equiv. to one of these.

- If $P$ (and so $B$) is completely regular, then $P \xrightarrow{p} B$ is locally trivial.

⇝ e.g. mflds, CW cpx, top. gps.

- Pullbacks of numerable $(\Pi,\Gamma)$-bundles along htpc maps are equivalent

### Interesting Examples

- algebraic + topological K-theory

# CLASSIFICATION

## §2.

<u>Recall</u>: $\cdot\ \mathcal{F}$ is a family of subgps closed under conjugacy (? + subgps) + $E\mathcal{F}$ is "universal $\mathcal{F}$-space"

$$E\mathcal{F} = \mathcal{H}^{!}\underline{\mathcal{F}} \text{ for } \mathcal{H}^{!} : \mathrm{Fun}(\mathcal{G}, \mathcal{U}) \to G\mathcal{U} \text{ and } \underline{\mathcal{F}} : h\mathcal{G} \to \mathrm{Set}$$

Elmendorf

$$G/H \mapsto \begin{cases} * & H \in \mathcal{F} \\ \emptyset & H \notin \mathcal{F} \end{cases}$$

- $E\mathcal{F} = E(\mathcal{F}|_H)$ as $H$-spaces, for $\mathcal{F}|_H = \{K \in \mathcal{F} \mid K \leq H\}$
- For $H \in \mathcal{F}$, $E(\mathcal{F})^H = E(\mathcal{F}^H)$ as $W_G H$-spaces, for $\mathcal{F}^H = \{K/H \mid K \in \mathcal{F} \text{ s.t. } H \leq K \leq N_G H\}$

<u>Defn</u> (Universal $(\Pi, \Gamma)$-bundle)

- Let $\mathcal{F}(\Pi, \Gamma) = \{\Lambda \leq \Gamma : \Lambda \cap \Pi = e\}$
  - $\rightsquigarrow$ $\mathcal{F}$-spaces are $\Pi$-free $\Gamma$-spaces
- Write $E(\Pi, \Gamma) = E\mathcal{F}(\Pi, \Gamma)$ and $B(\Pi, \Gamma) = E\mathcal{F}(\Pi, \Gamma)/\Pi$, so have principal $(\Pi, \Gamma)$-bundle $E(\Pi, \Gamma) \xrightarrow{\pi} B(\Pi, \Gamma)$
  - $\rightsquigarrow$ universal $\Pi$-free $\Gamma$-space $\quad\rightsquigarrow$ orbit space

<u>Note</u>: $B(\Pi, \Gamma)$ models $B\Pi$ as particular $G$-space

<u>Thm</u> - $E(\Pi, \Gamma) \to B(\Pi, \Gamma)$ is universal, i.e. there is a bijection

$$\left\{\begin{matrix} \text{equiv. classes} \\ \text{of principal} \\ (\Pi,\Gamma)\text{-bundles} \\ \text{over } X \end{matrix}\right\} \rightleftarrows \left\{\begin{matrix} G\text{-homotopy} \\ \text{classes of} \\ G\text{-maps} \\ X \to B(\Pi,\Gamma) \end{matrix}\right\}$$

i.e.

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E(\Pi,\Gamma) \arrow[d] \\ X \arrow[r] & B(\Pi,\Gamma) \end{tikzcd}$$

<u>Notation</u>: If $\Gamma = G \times \Pi$, write $E_G\Pi := E\mathcal{F}(\Pi, \Gamma)$ and $B_G\Pi := E\mathcal{F}(\Pi, \Gamma)/\Pi$.

# Examples of Classifying Spaces

§2.

**Goal**: Models for $E(\Pi,\Gamma)$ (and therefore models for $B(\Pi,\Gamma)$)

**Claim/Thm** - A principal $(\Pi,\Gamma)$-bundle $E$ is univ $\Leftrightarrow E^{\Lambda}\simeq *$ for all $\Lambda\leq\Gamma$ s.t. $\Lambda\cap\Pi=e$

① Let $Sec(EG,E\Gamma)\leq Map(EG,E\Gamma)$ in $\Gamma$-spaces (where $EG \circlearrowleft \Gamma$ by $q:\Gamma\to G$) be $\Gamma$-sections of $Eq: E\Gamma\to EG$.

**Thm**. This is model for univ. principal $(\Pi,\Gamma)$-bundle $\rightsquigarrow$ so $Sec(EG,E\Gamma)/\Pi$ is a model for $B(\Pi,\Gamma)$

② Other model: For $G$ discrete, $\Pi$ discrete or cpt Lie, $\Gamma=\Pi\rtimes G$

**Thm**. Guillou-May-Merling (2015): $BCat(\tilde{G},\tilde{\Pi})\to BCat(\tilde{G},\Pi)$ is universal

$\mapsto$ Ob $=G$ Mor $= {}_g\bullet \rightleftarrows \bullet_{g'}$

## Connecting $B_G\Pi$ and $B\Pi$

Simplify to $\Gamma=G\times\Pi$: In this case, $Sec(EG,E\Gamma)=Map(EG,E\Pi)$

so $Map(EG,E\Pi)/\Pi$ models $B_G\Pi$

Let $\mathcal{B}_G\Pi(X)=[X,B_G\Pi]_G$ and $\mathcal{B}\Pi(X)=[X,B\Pi]$

**Note**: By adjunction, a $G$-map $X\to Map(EG,B\Pi)\longleftrightarrow EG\times_G X\to B\Pi$

So if have $B_G\Pi \xrightarrow{\alpha} Map(EG,B\Pi)$, then get $\mathcal{B}_G\Pi(X)\to\mathcal{B}\Pi(EG\times_G X)$

$\parallel$

$Map(EG,E\Pi)/\Pi$ $\rightsquigarrow$ induced by $E(G\times\Pi)\to B(G\times\Pi)$

**Rmk**. More generally, $E\Gamma\to B\Gamma$ induces $\alpha: B(\Pi,\Gamma)\to Sec(EG,B\Gamma) = \left\{ \begin{tikzcd} EG \arrow[r, "f"] \arrow[dr] & B\Gamma \arrow[d, "Bq"] \\ & BG \end{tikzcd} \right\}$

**Q**. How much info does $\mathcal{B}_G\Pi(X)\to\mathcal{B}\Pi(EG\times_G X)$ lose?

**Thm(s)** - If $\Pi$ discrete, $\alpha$ is homeom. (If $\Gamma$ discrete, $\alpha$ homeom)

If $G$ cpt Lie and $\Pi$ Abelian cpt Lie, then $\alpha$ w.e.

# FIXED POINT THMS



Notation: Let $K \le G$, $\Lambda \le \Gamma$ s.t. $\Lambda \cap \Pi = e$ and $q : \Gamma \to G$ maps $q(\Lambda) \cong K$.
Set $\Pi^\Lambda = \Pi \cap N_\Gamma \Lambda = \Pi \cap Z_\Gamma \Lambda$

Rmk: $E(\Pi,\Gamma)^\Lambda = E(\mathcal{F}^\Lambda)$ for $\mathcal{F}^\Lambda = \{\Lambda'/\Lambda \mid \Lambda \le \Lambda' \le N_\Gamma \Lambda\}$ from earlier ch
is univ. principal $(\Pi^\Lambda, W_\Gamma\Lambda)$-bundle w/ base space $P(E(\Pi,\Gamma)^\Lambda)$

Thm - For $K \le G$, $B(\Pi,\Gamma)^K = \coprod_{\substack{\Pi\text{-conj. classes} \\ \text{of } \Lambda \text{ s.t.} \dots}} B(\Pi^\Lambda)$

As a $W_GK$-space:

$$= \coprod_{\substack{q^{-1}(N_KG)\text{-conj.} \\ \text{classes of } \Lambda \text{ s.t.} \dots}} W_GK \times_{V(\Lambda)} B(\Pi^\Lambda, W_\Gamma\Lambda)$$

$V(\Lambda)$: $\nwarrow W_\Gamma\Lambda/\Pi^\Lambda$

Pictorially

$$\begin{tikzcd} E(\Pi,\Gamma) \arrow[r, squiggly, "\Lambda\text{ f.p.}"] \arrow[d] & E(\Pi^\Lambda, W_\Gamma\Lambda) \arrow[d] \\ B(\Pi,\Gamma) \arrow[d, squiggly, "K\text{ f.p.}"] & B(\Pi^\Lambda, W_\Gamma\Lambda) \arrow[d, squiggly] \\ B(\Pi,\Gamma)^K = \coprod W_GK \times_{V(\Lambda)} & B(\Pi^\Lambda, W_\Gamma\Lambda) \end{tikzcd}$$

## Simplify to $\Gamma = G \times \Pi$

$\Lambda$ is of the form $\Lambda(\rho) = \{(h,\rho(h)) \mid h \in H \le G,\ \rho : H \to \Pi\}$
So can re-express thm in terms of $\rho : H \to \Pi$

Thm (again) - $B^H = \coprod_{[\rho] \in Rep(H,\Pi)} B(\Pi^{\Lambda(\rho)})$ $\leftarrow$ $Gp(H,\Pi)/\Pi$-conjugacy

In particular, $\Pi \to E^H \to p(E^H)$ is a principal $\Pi$-bundle

Rmk In general, $W_{G\times\Pi}\Lambda \not\cong V(\Lambda) \times \Pi^\Lambda$ so need general $1 \to \Pi \to \Gamma \to G \to 1$

Skipping: justification of
- Thm $H^*_G(B(\Pi,\Gamma)) \cong H^*(B\Gamma)$ w/ any coeff
- Borel $\nearrow$ $H^*_G(B(\Pi, G\times\Pi)) \cong H^*(BG) \otimes H^*(B\Pi)$ as $H^*(BG)$-module w/ field coeff

Q. What can we say about $\alpha^H : B(\Pi,\Gamma)^H \to Sec(EG, B\Gamma)^H$ ?

For $\Gamma = G \times \Pi$

$$B(\Pi,\Gamma)^H = \coprod_{[\rho] \in Rep(H,\Pi)} B(\Pi^{\Lambda(\rho)})$$
$$\downarrow \alpha^H$$
$$Sec(EG, B\Gamma)^H = Map(EG, BG \times B\Pi)^H = Map(EH, B\Pi)$$

$\rightsquigarrow$ restrict $\alpha^H$ to $B(\Pi^{\Lambda(\rho)})$:
adjoin to $B\nu : BH \times B(\Pi^{\Lambda(\rho)}) \to B\Pi$
where $\nu : H \times \Pi^{\Lambda(\rho)} \to \Pi$
$(h,\pi) \mapsto \rho(h)\pi$

THANKS FOR LISTENING :)

# Suggested Exercises

Just do the ones that look interesting to you!

## From May:

(1) Let $E \xrightarrow{p} B$ be a principal $(\Pi, \Gamma)$-bundle and $H \leq G$. Show $B^H$ is disjoint union of $p(E^\Lambda)$ for $\Lambda$ runs over $\Pi$-conjugacy classes of subgps $\Lambda \leq \Gamma$ s.t. $\Lambda \cap \Pi = e$ and $q : \Gamma \to G$ maps $q(\Lambda) \cong H$.

(2) Work out example 3.1 in the complex case

## Concrete Exercises:

(3) Stewart: For $n$ odd, have $SO(n) \times \{\pm 1\} \cong O(n)$. What are principal $(SO(n), O(n))$-bundles? How can we understand $E(SO(n), O(n)) \to B(SO(n), O(n))$ explicitly?

(4) Find an example $1 \to \Pi \to \Gamma \to G \to 1$ w/ $G$-space $B$ s.t. projection $B \times \Pi \to \Pi$ is <u>not</u> a trivial $(\Pi, \Gamma)$-bundle.

(5) Let $G = C_2$ and $\Pi = O(2)$, $\Gamma = G \times \Pi$. What is $B_{C_2} O(2)$? What is $B_{C_2} O(2)^{C_2}$?

## Open-ended questions: i.e. idk the answer or if these are good / interesting questions

(6) In the non-equivariant setting, a principal $\Pi$-bundle $E \xrightarrow{p} B$ is trivial iff it admits a section $s : B \to E$. This implies, e.g., that $E \times_B E \to E$ is a principal $\Pi$-bundle.

(a) Can you come up with a (partial) generalization of this to the equivariant setting?

(b) Or, characterize when the pullback $E \times_B E \to E$ is a principal $(\Pi, \Gamma)$-bundle?

(7) Some other non-equivariant stuff in equivariant setting:

(a) Let $\Pi^{\circlearrowleft G}$ and $\Pi' \leq \Pi$. When is $\Pi \to \Pi / \Pi'$ a principal $(\Pi, \Pi \rtimes G)$-bundle?

(b) Let $E \to B$ be a principal $(\Pi, \Gamma)$-bundle and $\Pi' \leq \Pi$. Under what conditions is $E \to E/\Pi'$ a principal $(\Pi', \Gamma)$-bundle?