Morse Theory + Flow Categories

- Write
- Say

Idea of MT is to study mflds by studying diff'l fns on them ↳ closed, f.d. Riemannian

Thm (CJS, 1990s) Let $f: M \to \mathbb{R}$ be a Morse fn on mfld M. Then (i) $B\mathcal{C}_f \simeq M$ and (ii) if f is Morse-Smale then $B\mathcal{C}_f \cong M$.

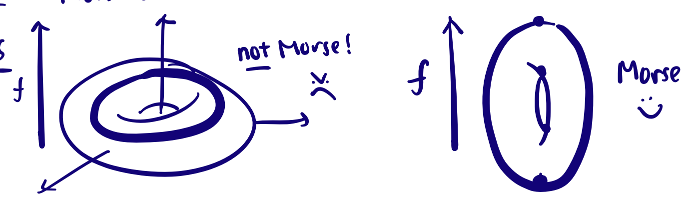
The nice fns we want to look at are called Morse fns

Defn - A fn $f: M \to \mathbb{R}$ is Morse if its critical pts are non-degenerate

↳ compact, f.d. Riemannian ↳ $df_p \equiv 0$ ↳ $d^2 f_p$ non-singular

Rmks Morse lemma ⇒ Crit(f) is discrete so M compact ⇒ finite

Exs

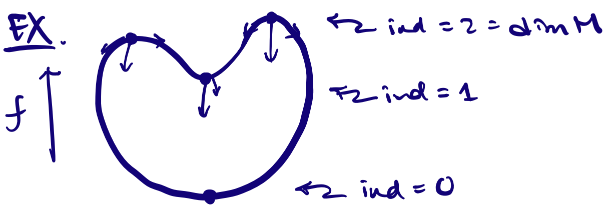


The Morse lemma also gives us another piece of info about the critical points, called the Morse index

Defn - The Morse index of $a \in \mathrm{Crit}(f)$ is $\mu(a) = $ index of $d^2 f_a$

captures idea of # of lin. indep. directions we can descend from a

Ex.



The next ingredient is the gradient flow, which gives us a way to "connect" points of differing indices

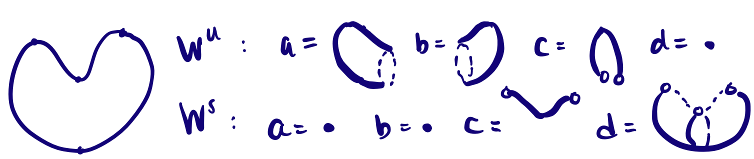
Defn - A flow line is an integral curve of $-\nabla f$ ← the gradient flow

Tell us how to "descend" along f. In our dented sphere example, like this...

Defn - $W^u(a) = \{\phi(t) : s(\phi) = a\}$ emanate from

$W^s(a) = \{\phi(t) : e(\phi) = a\}$ end up at

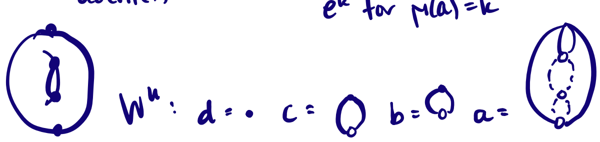
Ex.



Thm diffeom. to open disks w/ $\dim W^u(a) = \operatorname{codim} W^s(a) = \mu(a)$

Use to partition M: $M = \bigcup_{a \in \mathrm{Crit}(f)} W^u(a)$ ← CW decomp? e^k for $\mu(a) = k$

Not always: **Ex.**

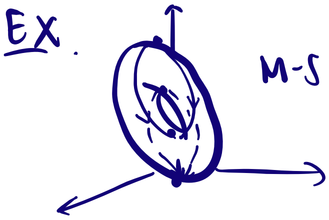


Have to attach 1-cell $W^u(b)$ in middle of 1-cell $W^u(c)$. ☹

Fix this: "Morse-Smale" condition on $f, -\nabla f$ — ask stable/unstable mflds to intersect transversely

One consequence is no flows b/w crit pts w/ same index.
So torus not M-S, but little deformation makes it MS

EX.



M-S! ☺

From here, usually talk about sublevel sets + Morse homology
But I want to talk about something a little different:

Flow Categories $\mathcal{C}_f$ capture similar info as in Morse homology but do different things with it

Ob: $\mathrm{Crit}(f)$

Mor: $\mathcal{C}_f(a,b) =$ "flows from a to b"

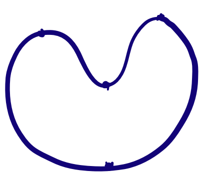
form $M(a,b) = W^u(a) \cap W^s(b) \,/\,$ flowing action — moduli space of flows

How to compose?

1. reparametrize (ok)
2. compactify (harder)

$\bar{M}(a,b) =$ moduli space of broken flows $a \to b =: \mathcal{C}_f(a,b)$

EX.

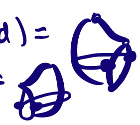


Ob = a, b, c, d

Mor: $\mathcal{C}_f(a,b) = \varnothing$ $\quad \mathcal{C}_f(c,d) = \{\}$

$\mathcal{C}_f(a,c) =$ $\quad \mathcal{C}_f(a,d) =$

$\mathcal{C}_f(b,c) =$ $\quad \mathcal{C}_f(b,d) =$



$\cong$ closed disks of dim $= \mu(a) - \mu(b)$

Thm (Cohen-Jones-Segal) $B\mathcal{C}_f \simeq M$ and if $(f, -\nabla f)$ Morse-Smale then $B\mathcal{C}_f \cong M$

What is $B\mathcal{C}_f$? Turns category into topological space

Defn The classifying space of category $\mathcal{C}$ is $B\mathcal{C}_f := |N\mathcal{C}_f|$ geometric realization of the nerve

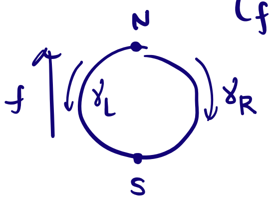
If this means nothing to you, don't worry about it.

roughly

$$B\mathcal{C}_f = \coprod_n \Delta^n \times (\text{n-composable morphisms}) \,/\, \begin{array}{l}\text{glue faces}\\ \text{collapse degens.}\end{array}$$

Let's look at an example to get the idea

Ex. S^1

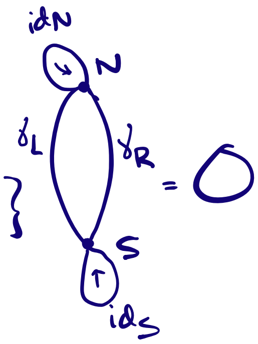


$\mathcal{C}_f$: $Ob = N, S$
$Mor = id_N, id_S, \gamma_L, \gamma_R$

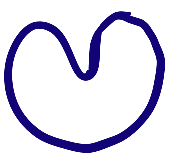
no interesting ways to compose

$\Rightarrow$ $B\mathcal{C}_f$:

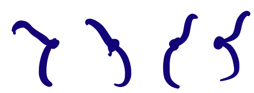
$\Delta^0 \times \{N, S\}$
$\Delta^1 \times \{id_N, id_S, \gamma_L, \gamma_R\}$



Ex.



$Ob = a, b, c, d$
$Mor =$ flows
$Mor \times_{Ob} Mor =$



$\Rightarrow$ $B\mathcal{C}_f$:

$\Delta^0 \times \{a, b, c, d\}$
$\Delta^1 \times Hom\,\mathcal{C}_f$
$\Delta^2 \times Hom \times_{Ob} Hom$

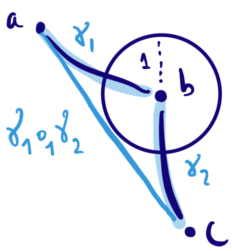
Also illustrate Thm!

Pf Sketch for (ii) / (i) still not proved — tried in my thesis but it didn't work. Still trying ☺

tecnical ♡ : $\exists$ assoc. gluing map $\mu : (0, \varepsilon] \times \mathcal{M}(a,b) \times \mathcal{M}(b,c) \to \mathcal{M}(a,c)$ for some $\varepsilon > 0$ (WLOG $\varepsilon = 1$)

Rmk Not published b/c "folk thm" that $\overline{\mathcal{M}}(a,b)$ mflds w/ corners + μ assoc.

Idea



$K(a,b) := \mathcal{M}(a,b) -$ "flows which get < 1 of other $c \in Crit(f)$"

Thm $K(a,b)$ compact and $\cong \overline{\mathcal{M}}(a,b)$

idea: form $B\mathcal{C}_f$ using $K(a,b)$ instead:

$$\coprod_{c_0 \to \cdots \to c_{k+1}} \underbrace{[f(c_{k+1}), f(c_0)] \times I^k}_{I^{k+1}} \times \big(K(c_0,c_1) \times \cdots \times K(c_k, c_{k+1})\big) /\sim$$

$\swarrow B\mathcal{C}_f$ $\quad$ $\swarrow \Delta^{k+1}$ $\quad$ $\cong \searrow$

$(t; s_1, \ldots, s_k; \gamma_0, \ldots, \gamma_k)$
$\downarrow ev$
$M \quad (\gamma_0 \circ_{s_1} \cdots \circ_{s_k} \gamma_k)(t)$

$P(I_f, M)$ CGWH?