Talbot 2022

Square K-theory & cut-and-paste manifolds

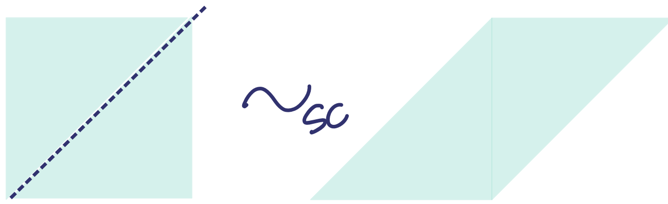
Maxine Elena Calle

Cut and paste invariants of manifolds via algebraic K-theory

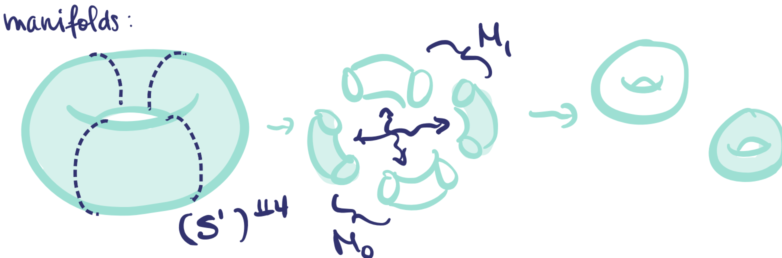
R. Hoekzema, M. Merling, L. Murray, C. Rovi, J. Semikina

SK-groups

for polytopes:



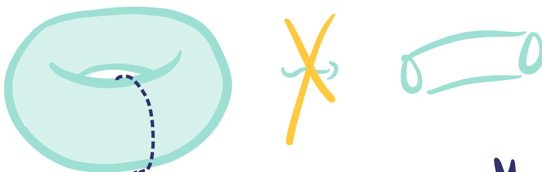
for manifolds:



SK-move

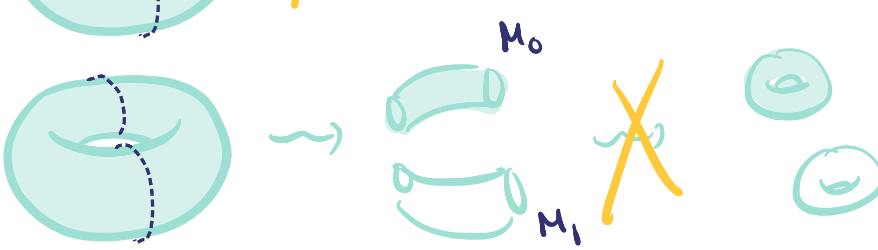
$T^2 \sim_{SK} T^2 \sqcup T^2$

Some non-examples:



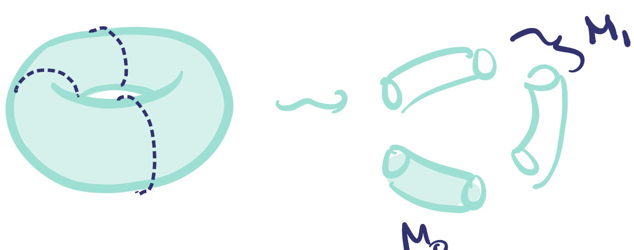
doesn't separate

$N \subseteq M, \quad M \setminus N \cong M_0 \sqcup M_1$



Can't glue ∂M_i to itself

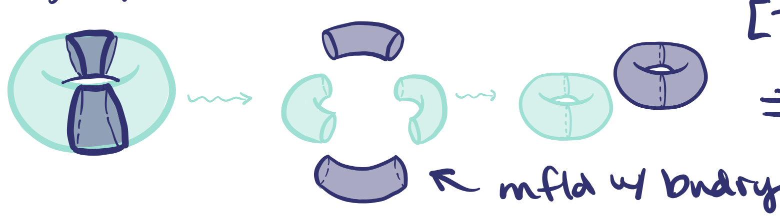
$\phi : N \to N$ diffeo



$\partial M_0 \not\cong \partial M_1 \not\cong N$

Defn - $SK_n = \mathbb{Z}[n\text{-mflds}] \Big/ \begin{matrix}\text{diffeo} \\ \sim SK\end{matrix}$ (or universal property defn)

Prop - It's a group!

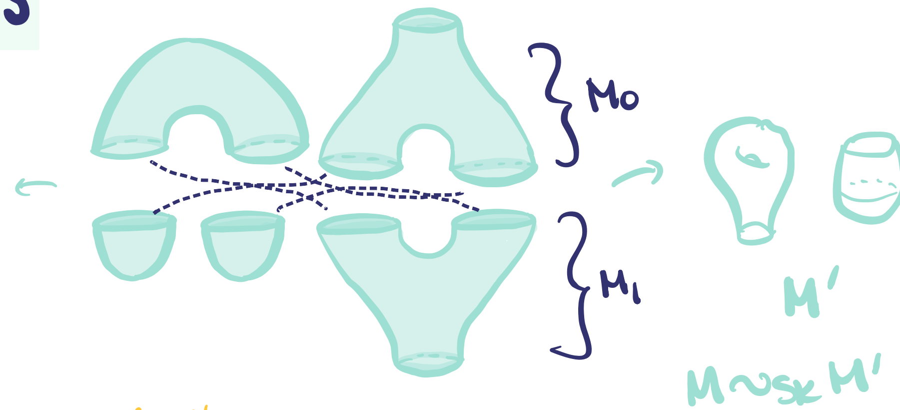


$[T^2] = 2[T^2]$

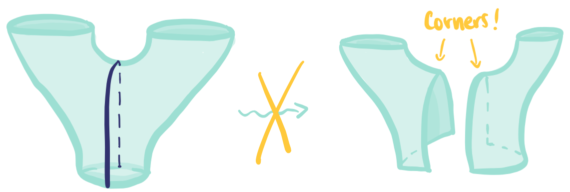
$\Rightarrow [T^2] = 0.$

SK^∂- groups

for manifolds w/ bndry:



<u>Non-examples</u>:



no cutting the bndry

cuts must be <u>reglued</u>

$\phi, \psi : N \to N$

$[M_0 \cup_\phi M_1] = [M_0 \cup_\psi M_1]$

gp completion

<u>Defn</u> - $SK^\partial_n := Gr\left(\mathbb{Z}[\text{n-mflds w/ bndry}] \Big/ \begin{matrix}\text{diffeo} \\ \sim_{SK^\partial}\end{matrix} \right)$

<u>Thm</u> (HMMRS) $SK^\partial_n \cong K_0^\square(Mfld^\partial_n)$ goal: understand this

<u>Rmks</u> - They also show $\exists$ SES

null bordant (n-1)-mfld / diffeo

$$0 \to SK_n \to SK^\partial_n \to C_{n-1} \to 0$$

$(M,\phi) \hookrightarrow$

$(M, \partial M) \mapsto \partial M$

$$\Rightarrow SK^\partial_n \cong SK_n \oplus C_{n-1}$$

$$SK_n \cong \begin{cases} 0 & n \text{ odd} \\ \mathbb{Z}[S^n] & n \equiv 2 \pmod 4 \\ \mathbb{Z}[S^n] \oplus \mathbb{Z}[\mathbb{CP}^{n/2}] & n \equiv 0 \pmod 4 \end{cases}$$

Square K-theory

exact categories	Waldhausen Categories	CGW Categories	Categories w/ Squares
Short exact sequences $0\to A\to B\to C\to 0$	Cofiber sequences $A\rightarrowtail B\twoheadrightarrow B/A$	Spans ⇝ squares $\begin{matrix}A & \twoheadrightarrow & B\\ \downarrow & \square & \downarrow\\ 0 & \rightarrowtail & C\end{matrix}$	squares $\begin{matrix}A & \rightarrowtail & B\\ \downarrow & \square & \downarrow\\ C & \rightarrowtail & D\end{matrix}$
Q	$S_\bullet$	Q	$C^{(\bullet)}$
in K_0: $[B]=[A]+[C]$	$[B]=[A]+[B/A]$	$[B]=[A]+[C]-[0]$	$[D]=[C]+[B]-[A]$

Defn – A category w/ squares consists of:

- a category $\mathcal{C}$ w/ coproducts
- a distinguished object 0
- subcategories of morphisms
  - $c\mathcal{C}$ "cofibrations" $\rightarrowtail$
  - $f\mathcal{C}$ "cofiber" $\twoheadrightarrow$
- distinguished squares $\square$

$$\begin{matrix}A & \rightarrowtail & B\\ \downarrow & \square & \downarrow\\ C & \rightarrowtail & D\end{matrix}$$

s.t. (i) $\square$ closed under coproduct
(ii) $\square$ commute, compose vertically + horizontally
(iii) $\mathrm{iso}\,\mathcal{C}\subseteq c\mathcal{C}, f\mathcal{C}$ and $\xrightarrow{\sim}$

$$\begin{matrix}A & \xrightarrow{\sim} & B\\ \downarrow & & \downarrow\\ C & \xrightarrow{\sim} & D\end{matrix}\qquad \begin{matrix}A & \to & B\\ \sim\downarrow & & \downarrow\sim\\ C & \to & D\end{matrix}$$ are in $\square$.

$$\begin{matrix}A\cap B & \hookrightarrow & B\\ \downarrow & & \downarrow\\ A & \hookrightarrow & A\cup B\end{matrix}$$

$|A\cup B|=|A|+|B|-|A\cap B|$

Mfld^∂_n

Ob = n-mflds w/ bndry
Mor = embeddings*
$0=\varnothing$
$c\mathcal{C}=f\mathcal{C}=$ all morphisms
$\square$ = pushouts

Defn – $\mathcal{C}^{(\bullet)}$ is simplicial category: $[n]\mapsto \mathcal{C}^{(n)}\subseteq \mathrm{Fun}([n],\mathcal{C})$

$\mathcal{C}^{(n)}$: Ob $=$ $C_0\twoheadrightarrow\cdots\twoheadrightarrow C_n$

Hom $=$ $\begin{matrix}C_0 & \twoheadrightarrow & \cdots & \twoheadrightarrow & C_n\\ \downarrow & \square & \downarrow \cdots \downarrow & \square & \downarrow\\ C_0' & \twoheadrightarrow & \cdots & \twoheadrightarrow & C_n'\end{matrix}$

No 1. fact about ssSets

$$|[n]\mapsto |N_n C^{(*)}||$$
$$\cong |[m]\mapsto |N_* C^{(m)}||$$
$$\cong |[n]\mapsto N_n C^{(n)}|$$

Then $N_*\mathcal{C}^{(*)}$ is a bisimplicial set.

Defn $K^\square(\mathcal{C}):=\Omega|N_*C^{(*)}|$ and $K^\square_i(\mathcal{C}):=\pi_i K^\square(\mathcal{C})$.

Ex. Waldhausen categories

(1) $c\mathcal{C}=co\mathcal{C}$
$f\mathcal{C}$ = cofiber (quotient)
$\square$ = all comm.
$0=0$
☆ when w.e. $=\mathrm{iso}\,\mathcal{C}$ ☆

(2) $c\mathcal{C}=co\mathcal{C}$
$f\mathcal{C}$ = all
$\square$ = pushouts (up to w.e.)
$0=0$
(always works)

$\Longrightarrow K^\square(\mathcal{C})\simeq K^W(\mathcal{C})$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\ C \arrow[r] \arrow[drr, bend right] & C\cup_A B \arrow[dr, "\text{w.e.}"] & \\ & & D \end{tikzcd}$$

$K_0^{\square}(\mathrm{Mfld}_n^{\partial})$

<u>Thm</u> $K_0^{\square}(\mathcal{C}) \cong \mathbb{Z}[\mathrm{Ob}\,\mathcal{C}] \Big/ \begin{matrix} [0]=0 \\ [A]+[D]=[B]+[C] \text{ for all } \end{matrix}$ $\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & B \arrow[d, tail] \\ C \arrow[r, tail] & D \end{tikzcd}$

if
- 0 is initial <u>or</u> terminal in $\mathcal{C}$, $\downarrow\mathcal{C}$
- for all $A, B \in \mathcal{C}$ $\exists X \in \mathcal{C}$ s.t.

$\exists$ $\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & A \arrow[d, tail] \\ B \arrow[r, tail] & X \end{tikzcd}$ $\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & B \arrow[d, tail] \\ A \arrow[r, tail] & X \end{tikzcd}$

in $\mathrm{Mfld}_n^{\partial}$,
$\emptyset$ is initial
$X = A \sqcup B$

<u>Cor</u> $K_0^{\square}(\mathrm{Mfld}_n^{\partial}) \cong \mathbb{Z}\begin{bmatrix} n\text{-mflds} \\ \text{w/ bndry} \end{bmatrix} \Big/ \begin{matrix} [\emptyset]=0 \\ [M \cup_N M'] = [M]+[M']-[N] \end{matrix}$

$$\boxed{K_0^{\square}(\mathrm{Mfld}_n^{\partial}) \cong SK_n^{\partial}}$$

Pf idea: same gens & relation

<u>Thm</u>
Euler characteristic: $SK_n^{\partial} \xrightarrow{\chi} \mathbb{Z}$ lifts to map $K^{\square}(\mathrm{Mfld}_n^{\partial}) \to K(\mathbb{Z})$.

induced by $\mathrm{Mfld}_n^{\partial} \xrightarrow{S} \mathrm{Ch}_{\mathbb{Z}}^{hb}$ singular chains