BUGCAT
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Equivariant Partition Complexes + Trees

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joint work w/ Julie Bergner, Peter Bonventre, David Chan, + Maru Sarazola

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Partition Complex

Given a finite set

$$\underline{n} = \{1, 2, \ldots, n\},$$

we can <u>partition</u> it in different ways,

e.g. $n = 6$

$$\begin{tikzcd}
 & (12)(34)(56) \arrow[dr] & \\
(1)(2)(34)(5)(6) \arrow[ur] \arrow[dr] & & (12)(3456) \\
 & (12)(346)(5) \arrow[ur] &
\end{tikzcd}$$

and the set of partitions forms a poset $P(\underline{n})$ under coarsening.

Defn - The <u>partition complex</u> is the classifying space of this poset category, $BP(\underline{n})$.

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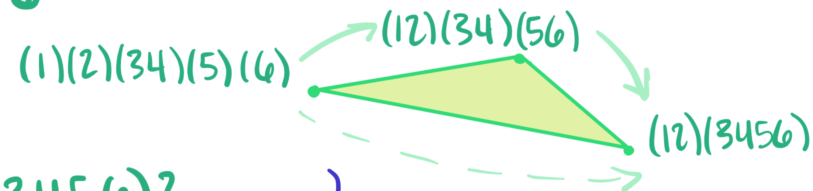
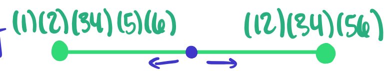


Note (123456)?
(1)(2)(3)(4)(5)(6)? } remove these to make it interesting

Q. How to think about

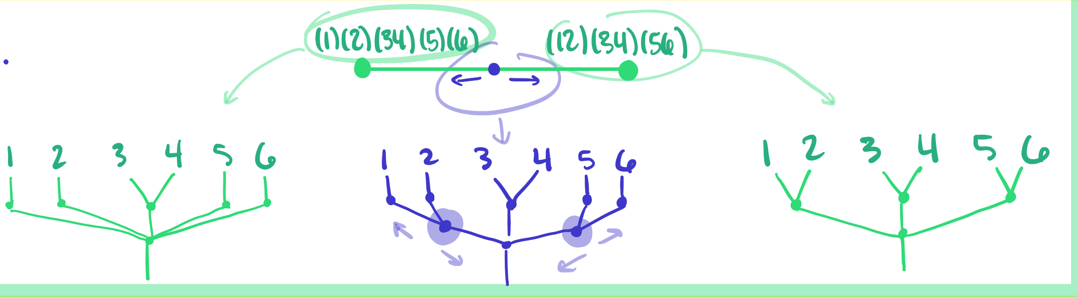


as a "deformation" of one partition to another?

A. Trees!

Trees

Idea.



More formally: ① functor $P(\underline{n}) \to T(\underline{n})$ ← category of trees
(Heuts-Moerdijk, 2021)

② map $BP(\underline{n}) \to \mathbb{T}(\underline{n})$ ← Space of "measured" trees
(Robinson, 2004)

category of "layered" trees

Thm. (①+②). There are zig-zags $BP(n) \xleftarrow{\sim} B\Delta P(\underline{n}) \xrightarrow{\sim} BT(\underline{n})$ from functors ①

$BP(\underline{n}) \xleftarrow{\cong} \mathbb{T}(\underline{n}) \xrightarrow{\cong} BT(\underline{n})$ not from functors ②

+ more ...

But what if there's a group action?

enter: equivariant homotopy theory

G - finite group

$G \curvearrowright A$ ← means we have a map $G \times A \xrightarrow{\alpha} A$

$(g, a) \mapsto g \cdot a$

e.g. $G = C_2 = \{\pm 1\}$

$A = \{x, -x, y, -y, z\} = C_2/e \sqcup C_2/e \sqcup C_2/C_2$

$\underline{n} = 1 \ 2 \ \cdots \ n$ trivial action

Q. What is an "equivariant partition"?

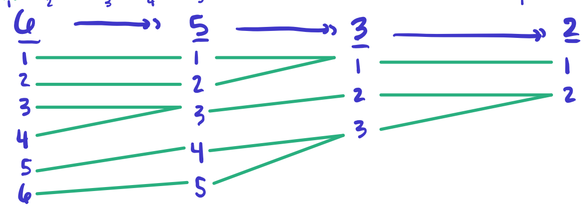
How much of the non-equivariant story holds?

Equivariant Partitions

① reinterpret "partition"

e.g. $n=6$

$(1)_1(2)_2(34)_3(5)_4(6)_5 \to (12)_1(34)_2(56)_3 \to (12)_1(3456)_2$



Partition of A ⟷ surjective map $A \twoheadrightarrow \underline{n}/\sim$

n = # of parentheses pairs

Partition of $A^{\circlearrowleft G}$ ⟷ Surjective map $A \twoheadrightarrow ?$

② a few directions...

Naive	Weird	Fully
${}_{G}\circlearrowright A \twoheadrightarrow \underline{n}/\sim$ non-equivar	${}_{G}\circlearrowright A \twoheadrightarrow B^{\circlearrowleft G}/\sim$ non-equivar	$A^{\circlearrowleft G} \twoheadrightarrow B^{\circlearrowleft G}/\sim$ equivar
$\begin{tikzcd} A \arrow[r, two heads] \arrow[dr, dashed, two heads] & \underline{n} \arrow[d] \\ & \underline{m} \end{tikzcd}$	$\begin{tikzcd} A \arrow[r, two heads] \arrow[dr, dashed, two heads] & B \arrow[d] \\ & B' \end{tikzcd}$	$\begin{tikzcd} A \arrow[r, two heads] \arrow[dr, dashed, two heads] & B \arrow[d] \\ & B' \end{tikzcd}$
$\mathcal{P}(A)$	$\mathcal{P}_G(A)$	$\mathcal{P}^G(A)$
$\mathcal{T}(A)$	$\mathcal{T}_G(A)$??	$\mathcal{T}^G(A)$

Interactions:

$$\mathcal{P}(A)^{\circlearrowleft G} \hookrightarrow \mathcal{P}_G(A)^{\circlearrowleft G} \hookleftarrow \mathcal{P}^G(A)$$

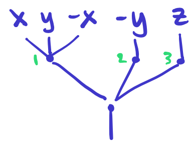
$$\mathcal{P}(A)^H \simeq \mathcal{P}^H(\downarrow^G_H A)$$

... and their trees

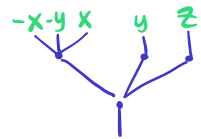
Example $G = C_2$, $A = C_2/e \sqcup C_2/e \sqcup C_2/C_2$

$x \rightleftarrows -x \qquad y \rightleftarrows -y \qquad z \circlearrowleft$

(naïve) $P(A)$: $A \twoheadrightarrow \underline{3}$ $\quad (x\,y\,{-x})_1(-y)_2(z)_3 \overset{g}{\longmapsto} (-x\,{-y}\,x)_1(y)_2(z)_3$



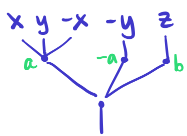
$\longmapsto$



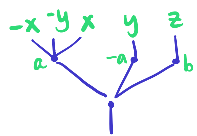
(weird) $P_G(A)$: $(x\,y\,{-x})_a(-y)_{-a}(z)_b \overset{g}{\longmapsto} (-x\,{-y}\,x)_{-a}(y)_a(z)_b \sim (-x\,{-y}\,x)_a(y)_{-a}(z)_b$

$\underline{3}^{\circlearrowleft C_2} = C_2/e \sqcup C_2/C_2$

$a \rightleftarrows -a \qquad b \circlearrowleft$



$\longmapsto$

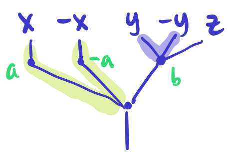


$\circlearrowleft G$ $\circlearrowleft G$ ← independent

(fully) $P^G(A)$: $(x)_a(-x)_{-a}(y\,{-y}\,z)_b \longmapsto (-x)_{-a}(x)_a(-y\,y\,z)_b$

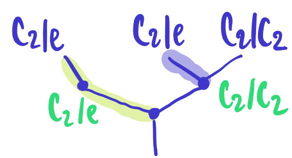
equivariant

$\pm x \to \pm a$
$\pm y \to b$
$z \to b$



$\circlearrowleft G$

"orbits are ⇝ points"



"G-trees"

Main Results

Thms (BBCCS). There are zig-zags

$$BP(A) \xleftarrow{\simeq_G} B\Delta P(A) \xrightarrow{\simeq_G} BT(A)$$

G-cat of "layered" trees; G-cat of trees; from G-functors

$$BP(A) \xleftarrow{\simeq_G} \mathbb{T}(A) \xrightarrow{\simeq_G} BT(A)$$

not from G-functors; G-space of "measured" trees

and same for $\mathcal{P}^G(A)$.

- Homotopy type: $\mathcal{P}(A)^G \simeq \begin{cases} \bigvee \underbrace{SP(G/H)}_{H<K<G} \wedge \underbrace{SP(\underline{n})}_{\bigvee S^{n-3}} & \text{if } A = \coprod_{i=1}^{n} G/H, \\ * & \text{else.} \end{cases}$
- Homology: $H^{n-3}(\Pi(A)) \cong \varepsilon^G \otimes \mathrm{Lie}_A$

  integral sign representation

Future Directions - Tits buildings? operads?? Goodwillie calculus?!?

Thanks for listening!

References

"Equivariant Trees + Partition Complexes" Bergner - Bonventre - C.-Chan-Sarazola

"Partition Complexes + Trees" Heuts - Moerdijk

"Partition complexes, duality, and integral tree representations" Robinson