The Parametrized h-Cobordism Theorem

The K-theory Part

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Recall:

The s-Cobordism Theorem

$$\left\{ \begin{array}{c} h\text{-cobordisms} \\ M \hookrightarrow W \hookleftarrow M' \end{array} \right\}\Big/\sim \ \underset{\text{bijection}}{\cong} \ K_1(\mathbb{Z}[\pi_1 M])/\sim$$

Connor's talk $\rightsquigarrow$ $\mathcal{H}^\infty(M)$ space of h-cobs $\xrightarrow{\pi_0}$

$\longleftrightarrow$ param. h-cob

my talk $\rightsquigarrow$?

Q: How is $K_1(\mathbb{Z}[\pi_1 M])$ a π_0 construction?

Recall $\pi_1 M \cong \pi_0 \Omega M$, so $\pi_1 K(\mathbb{Z}[\pi_1 M]) \cong \pi_0 \Omega K(\mathbb{Z}[\pi_0 \Omega M])$

$\rightsquigarrow K(\underbrace{\mathbb{Z}[\Omega M]}_{Ch(\Omega M)})$? Not quite: $K(\mathbb{S}[\Omega X]) = K(\mathrm{Mod}_{\Sigma^\infty_+ \Omega X})$

$\simeq A(X)$ Waldhausen's algebraic KT of spaces

Algebraic K-Theory

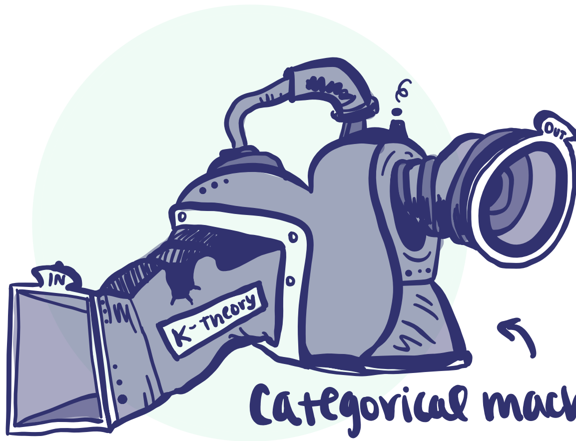
Idea:

Category $\mathcal{C}$ with "extra structure"

Sym mod: $X \otimes Y$

exact: $0 \to A \hookrightarrow B \twoheadrightarrow C \to 0$

Waldhausen: $A \rightarrowtail B \twoheadrightarrow B/A$



Categorical machinery

$K(\mathcal{C})$

∞-loop space

$$\downarrow \pi_n$$

$K_n(\mathcal{C}) = \pi_n K(\mathcal{C})$

Characterizing property:

- $K_0(\mathcal{C}) \cong \mathbb{Z}[\mathrm{ob}\,\mathcal{C}] \,/\, [B] = [A] + [B/A]$
- Additivity Theorem lifts K_0-reln to ∞-loop spaces
- "universal additive invariant" formalized in ∞-categories

Algebraic K-Theory of Spaces

Defn: the category of retractive spaces

R_X

$$\begin{tikzcd} Y \arrow[r] \arrow[d, bend right, "r"'] & Y' \arrow[d, bend right, "r'"'] \\ X \arrow[u, bend right, "i"'] \arrow[r, equal] & X \arrow[u, bend right, "i'"'] \end{tikzcd}$$

$r \circ i \simeq id_X$

cofiber sequences

$Y \rightarrowtail Y' \twoheadrightarrow Y' \cup_Y X$

Ex $X = *$

R_* $\quad \exists! \begin{tikzcd} Y \arrow[d, bend right] \\ * \arrow[u, bend right, "i"'] \end{tikzcd} = S_*$

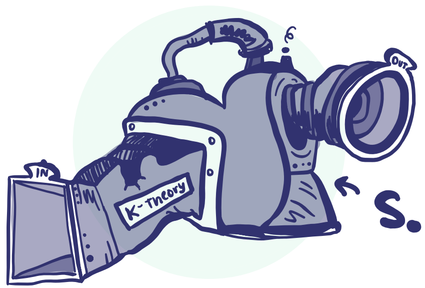
$Y \rightarrowtail Y' \twoheadrightarrow Y'/Y$

Ex. R_*^{hf} = finite CW cpxs (up to htpy)

Rmk: $\begin{tikzcd} Y \arrow[d, bend right, "r"'] \\ X \arrow[u, bend right, "i"'] \end{tikzcd} \rightsquigarrow Y$ w/ ΩX-action (just like $\mathbb{Z}[\pi_1 X] \circlearrowright Ch_*(Y,X)$)

finiteness condition ≈ rel CW

$R_X^{hf} \rightarrow$



$\rightarrow K(R_X^{hf}) =: A(X)$

$S_\bullet$-construction

Stable Parametrized h-Cobordism theorem

If M is a smooth manifold, there are maps of ∞-loop spaces

$$\begin{tikzcd} \mathcal{H}^\infty(M) \arrow[r] & \underbrace{\Sigma^\infty_+ M}_{\text{stable homotopy}} \arrow[r] & A(M) \arrow[r] \arrow[l, bend right, "\text{splits}"'] & \underbrace{Wh(M)}_{\text{Whitehead spectrum}} \end{tikzcd}$$

which is a fiber sequence up to homotopy.

- $\Omega Wh(M) \simeq \mathcal{H}^\infty(M)$ sometimes different conventions
- $A(M) \simeq \Sigma^\infty_+ M \times Wh(M)$

Rmk: In Top or PL settings,

$$\mathcal{H}^\infty(M) \to A(*) \wedge \Sigma^\infty_+ M \to A(M) \to Wh(M)$$

Back to the s-Cobordism theorem

Q: How does this relate back to $\pi_0 \mathcal{H}^\infty(M) \cong K_1(\mathbb{Z}[\pi_1 M])$?
What is "higher Whitehead torsion"?

There is a miraculous "linearization map"

$$A(X) \xrightarrow{L} K(\mathbb{Z}[\pi_1 X])$$

— build using perfect $\mathbb{Z}[\pi_1 X]$ chain cpxs

e.g. if X simply cnt'd, $Y \mapsto Ch_*(Y, X)$

which is an iso on π_0, π_1.

In particular,

$$\begin{array}{ccc} A_1(X) & \xrightarrow{\cong} & K_1(\mathbb{Z}[\pi_1 X]) \\ \downarrow & & \downarrow \\ A_1(X)/\pi_1^s(X) & \cong & Wh_1(X) \end{array}$$

Slogan: Higher Whitehead Torsion $\in Wh(X) \simeq A(X)/$ stable htpy of X