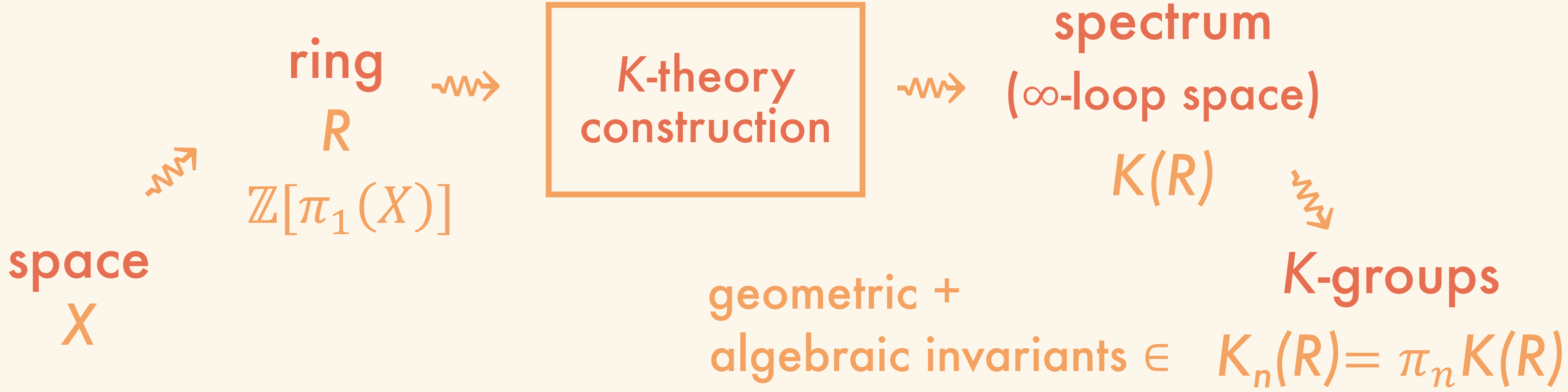
JMM 2024

a linearization map for equivariant algebraic K-theory of spaces

Maxine E. Calle
with David Chan and Andres Mejia

What is algebraic *K*-theory?



space X

ring R

$\mathbb{Z}[\pi_1(X)]$

K-theory construction

spectrum (∞-loop space) $K(R)$

K-groups

geometric + algebraic invariants ∈ $K_n(R) = \pi_n K(R)$

Higher algebraic *K*-theory captures *geometric information* via $\mathbb{Z}[\pi_1(X)]$

A-theory

Higher algebraic *K*-theory captures *geometric information* via $\mathbb{Z}[\pi_1(X)]$

Wall finiteness obstruction

$K_0(\mathbb{Z}[\pi_1(X)])/\sim$

Whitehead torsion

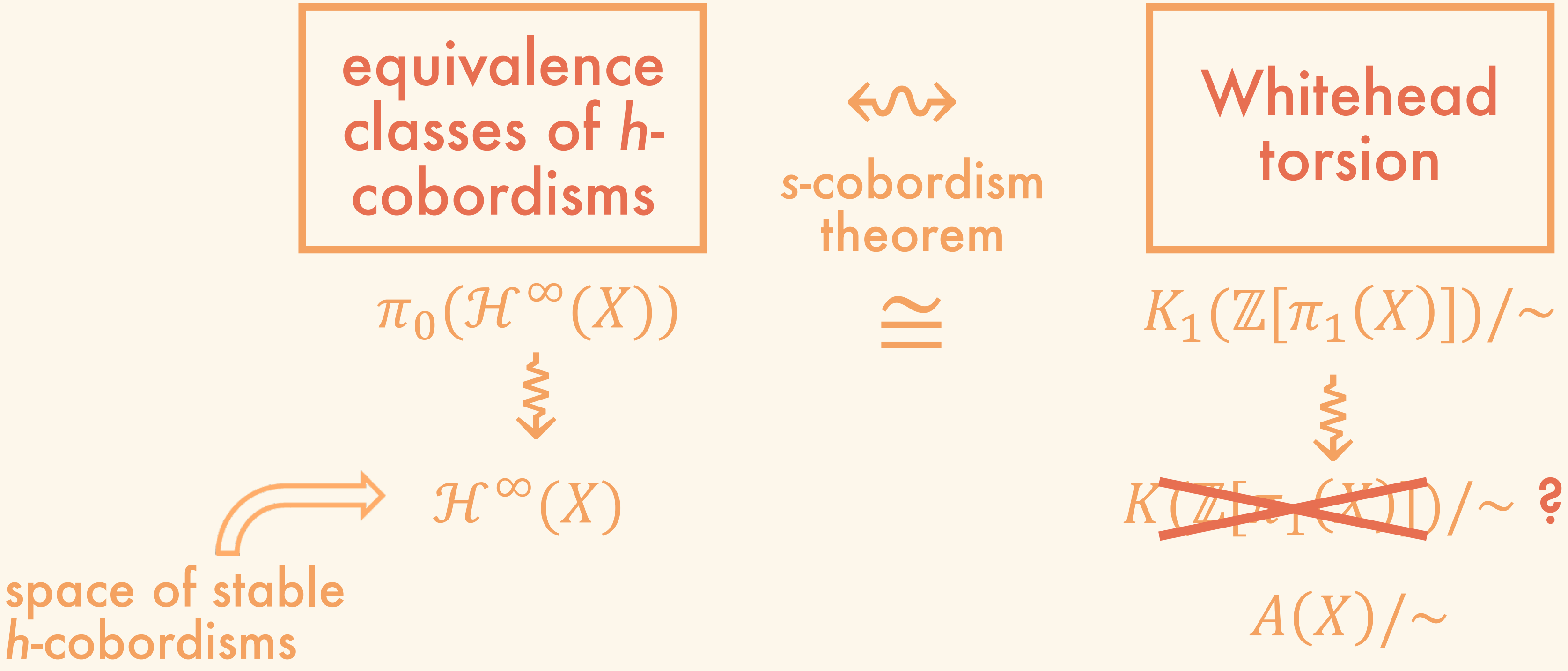
$K_1(\mathbb{Z}[\pi_1(X)])/\sim$

But $\mathbb{Z}[\pi_1(X)]$ can't see everything...

A-theory

Higher algebraic *K*-theory captures *geometric information* via $\mathbb{Z}[\pi_1(X)]$

But $\mathbb{Z}[\pi_1(X)]$ can't see everything...



equivalence classes of *h*-cobordisms

s-cobordism theorem

Whitehead torsion

$$\begin{tikzcd}
\pi_0(\mathcal{H}^\infty(X)) \arrow[r, phantom, "\cong"] \arrow[d, squiggly] & K_1(\mathbb{Z}[\pi_1(X)])/\sim \arrow[d, squiggly] \\
\mathcal{H}^\infty(X) & K\xcancel{(\mathbb{Z}[\pi_1(X)])}/\sim \ ?
\end{tikzcd}$$

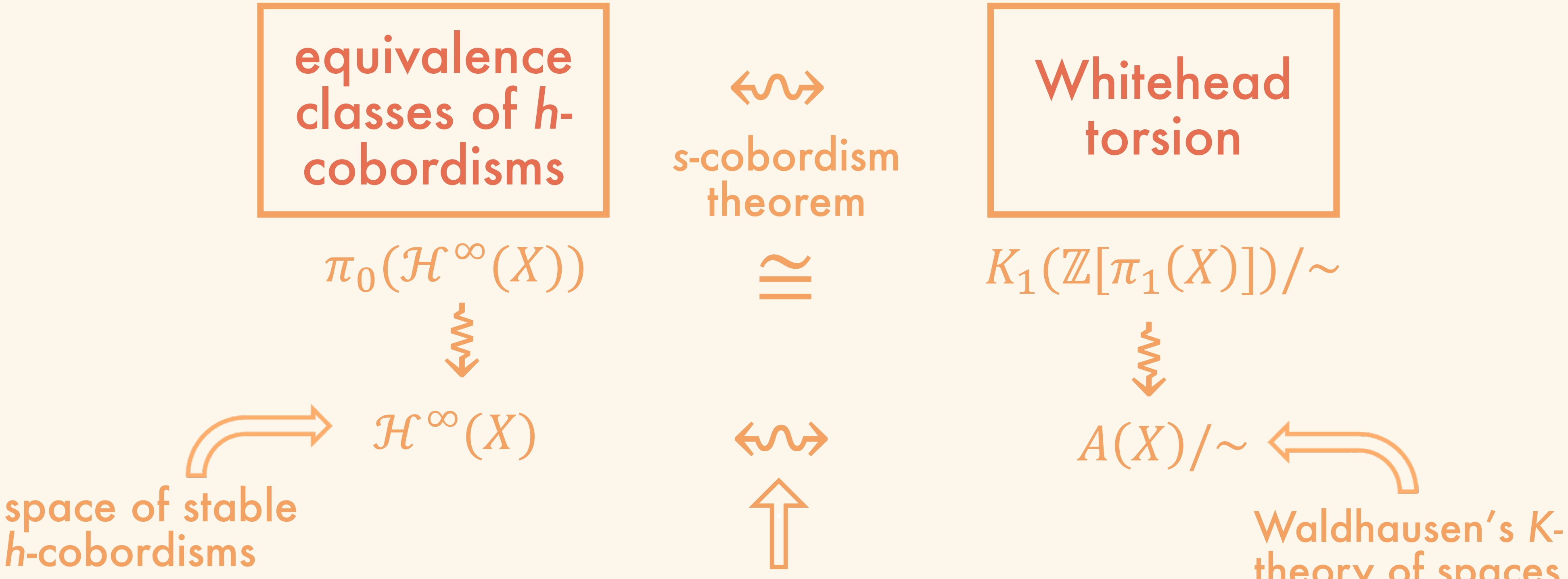
$A(X)/\sim$

space of stable *h*-cobordisms

A-theory

Higher algebraic *K*-theory captures *geometric information* via $\mathbb{Z}[\pi_1(X)]$

But $\mathbb{Z}[\pi_1(X)]$ can't see everything...



$$\begin{tikzcd}
\boxed{\text{equivalence classes of } h\text{-cobordisms}} \arrow[r, leftrightsquigarrow, "\text{s-cobordism theorem}"'] & \boxed{\text{Whitehead torsion}} \\
\pi_0(\mathcal{H}^\infty(X)) \arrow[d, rightsquigarrow] \arrow[r, phantom, "\cong"] & K_1(\mathbb{Z}[\pi_1(X)])/\sim \arrow[d, rightsquigarrow] \\
\mathcal{H}^\infty(X) \arrow[r, leftrightsquigarrow] & A(X)/\sim
\end{tikzcd}$$

space of stable *h*-cobordisms

Waldhausen's *K*-theory of spaces

The parametrized stable *h*-cobordism theorem

Jahren-Rognes-Waldhausen (2013)

A-theory

Higher algebraic *K*-theory captures
geometric infor via $\mathbb{Z}[\pi_1(X)]$

But $\mathbb{Z}[\pi_1(X)]$

sp
h-cobordisms

The parametrized stable *h*-c

Jahren-Rognes-Waldha (20

Wait...

What's the connection between $A(X)$ and $K(\mathbb{Z}[\pi_1(X)])$?

How much information does $K(\mathbb{Z}[\pi_1(X)])$ capture?

2-connected

The amazing *linearization map*

$$\ell\colon A(X) \to K(\mathbb{Z}[\pi_1(X)])$$

A_G-theory

What about equivariantly?

equivariant input $\rightsquigarrow$ **K-theory construction** $\rightsquigarrow$ equivariant output

$X \circlearrowright G$ (finite group) $\rightsquigarrow$ $A_G(X)$

Malkiewich-Merling

$A_G(X) \Longleftarrow$ built from retractive G-spaces

A_G-theory

What about equivariantly?

$X \circlearrowright G$ (finite group) $\overset{\text{Malkiewich-Merling}}{\rightsquigarrow}$ $A_G(X)$

Theorem. If X is a G-space, $A_G(X)$ is a genuine G-spectrum which fits into an equivariant stable parametrized *h*-cobordism theorem and whose fixed points split

Malkiewich-Merling (2010s)

$$A_G(X)^G \xrightarrow{\sim} \prod_{(H)\leq G} A(X^H_{hWH})$$

Badzioch-Dorabiala (2017)

Question. Does $A_G(X)$ admit a linearization map?

What's the equivariant generalization of $K(\mathbb{Z}[\pi_1(X)])$?

A_G-theory

What about equivariantly?

What's the equivariant generalization of $K(\mathbb{Z}[\pi_1(X)])$?

$$\mathbb{Z}[\pi_1(X)] \circlearrowright G$$

$$K(\mathbb{Z}[\pi_1(X)]) \circlearrowright G$$

~~naïve G-spectrum~~

genuine G-spectrum

$$K_\theta(\mathbb{Z}[\pi_1(X)])$$

equivariant algebraic *K*-theory
of G-rings (Merling)

fixed points capture

$$\cancel{\mathbb{Z}[\pi_1(X)]_\theta^H}$$

$$\mathbb{Z}[\pi_1(X^H_{hWH})]$$

☺

$$K_G(\underline{\mathbb{Z}[\pi_1(X)]})$$

K-theory of ring coefficient
systems (C.-Chan-Mejia)

A_G-theory

What about equivariantly?

What's the equivariant generalization of $K(\mathbb{Z}[\pi_1(X)])$?

$K_G(\underline{\mathbb{Z}[\pi_1(X)]})$ genuine G-spectrum

$$X \circlearrowright G \rightsquigarrow \cancel{\mathbb{Z}[\pi_1(X)]} \circlearrowright G$$

$\underline{\mathbb{Z}[\pi_1(X)]}$ coefficient system of rings

$G/H \mapsto \mathbb{Z}[\pi_1(X^H)]$

with restriction maps

A_G-theory

What about equivariantly?

What's the equivariant generalization of $K(\mathbb{Z}[\pi_1(X)])$?

$K_G(\underline{\mathbb{Z}[\pi_1(X)]})$ genuine G-spectrum

$X \circlearrowleft G \rightsquigarrow \underline{\mathbb{Z}[\pi_1(X)]}$ coefficient system of rings $\rightsquigarrow K_G(\underline{\mathbb{Z}[\pi_1(X)]})^G$

$G/H \mapsto \mathbb{Z}[\pi_1(X^H)]$
with restriction maps

$\simeq \prod_{(H)\leq G} K(\mathbb{Z}[\pi_1(X^H_{hWH})])$

$\Uparrow$

theorem from
David's talk

Main Theorem. There is a genuine equivariant linearization map

C.-Chan-Mejia (2023)

$$L\colon A_G(X) \to K_G(\underline{\mathbb{Z}[\pi_1(X)]})$$

main thm

There is a map of spectral Mackey functors

$$A_G(X) \xrightarrow{L} K_G(\underline{\mathbb{Z}[\pi_1(X)]})$$

which splits on fixed points

$$\begin{tikzcd}
A_G(X)^G \arrow[r, "L^G"] \arrow[d, "\sim"'] \arrow[dr, phantom, "\circlearrowright"] & K_G\left(\underline{\mathbb{Z}[\pi_1(X)]}\right)^G \arrow[d, "\sim"] \\
\prod_{(H)\leq G} A(X^H_{hWH}) \arrow[r, "\prod \ell"'] & \prod_{(H)\leq G} K(\mathbb{Z}[\pi_1(X^H_{hWH})])
\end{tikzcd}$$

where ℓ is non-equivariant linearization

Consequences.

- L is a 2-connected map of genuine G-spectra
- Can lift invariants from $K_G(\underline{\mathbb{Z}[\pi_1(X)]})$ to $A_G(X)$
- Future work (see Andres's talk!)

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