Stochastic Impatience and the Separation of Time and Risk Preferences[*]

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Abstract

We study how the separation between time and risk preferences relates to a behavioral property that generalizes impatience to stochastic environments: Stochastic Impatience. We show that Stochastic Impatience holds if and only if risk aversion is "not too high" relative to the inverse elasticity of intertemporal substitution. This result has implications for many known models. For example, in the models of Epstein and Zin (1989) and Hansen and Sargent (1995), Stochastic Impatience is violated for all commonly used parameters. If Stochastic Impatience is taken normatively, this suggests a limit on the amount of separation between time and risk preference; otherwise, it provides a simple one-question test for it.

Key words: Stochastic Impatience, Epstein and Zin preferences, Separation of Time and Risk preferences, Risk Sensitive preferences, Non-Expected Utility

JEL: D81, D90, G11, E7.

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1   Introduction

In the standard Expected Discounted Utility model, the inverse of the elasticity of intertemporal substitution (EIS) is equal to the coefficient of relative risk aversion. However, an enormous literature in macroeconomics, finance, and behavioral economics has pointed out to the need for separating these two coefficients both on empirical and on conceptual grounds. Empirically, observations from lab experiments, longitudinal micro-data, and the desire to fit macroeconomic and financial data require a higher coefficient of risk aversion than the inverse of EIS.[1] Conceptually, attitudes towards risk and towards intertemporal smoothing belong to different domains, and there is no compelling reason why they should be equal to each other. These observations have led to the development of models that separate risk attitudes from EIS, with the CRRA-CES version of Epstein and Zin (1989) (henceforth EZ) and the Risk-Sensitive preferences of Hansen and Sargent (1995) (henceforth HS) being the most prominent examples. Given the fundamental role of risk aversion and EIS in economics, evaluating these models qualitatively and quantitatively is an issue of primary relevance in the discipline.

In this paper, we show that a behavioral postulate that we call *Stochastic Impatience* imposes a bound on how high risk aversion can be relative to EIS. Consider the choice between the following two options:

A. With equal probability, permanently increase consumption by either 20% starting today, or by 10% starting next year;

B. With equal probability, permanently increase consumption by either 10% starting today, or by 20% starting next year.

Both options involve identical benefits, odds, and dates. However, in option A the highest increase (20%) is paired with the earlier date, whereas in B it is paired with the later date. What would, or should, an individual choose?

To the extent that an individual prefers higher payments sooner, it is plausible that she picks option A. One way to see this is by decomposing each alternative into two parts. Both A and B offer a basic lottery in which the individual receives an increase of 10% either today or next year, as well as a 50-50 chance of an additional increase of 10%. The only difference between them is when this additional payment is made: today in option A, next year in option B. Insofar as the individual prefers obtaining a payment sooner, option A should be preferred. Since this property is a version of Impatience (a preference for earlier payments) for risky environments, we call it Stochastic Impatience. Impatience and Stochastic Impatience are equivalent under Expected Discounted Utility, but when time and risk preferences are separate,

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[1]For example, Barsky et al. (1997) study a large cross section of American households and find that risk aversion and EIS are uncorrelated. See also Bansal and Yaron (2004); Hansen et al. (2007); Barro (2009); Andreoni and Sprenger (2012); Nakamura et al. (2017) and references therein.

this is no longer true. Our main result shows that under general conditions, Stochastic Impatience is violated whenever risk aversion is high enough for a fixed EIS.

We begin our analysis by considering the widely used CRRA-CES version of EZ preferences. We show that Stochastic Impatience fails if the coefficient of risk aversion is both above the inverse of EIS and above one. All applications of EZ that we are aware of use parameters in this range: assuming that risk aversion is above the inverse of EIS is the primary reason to use EZ in the first place. For example, with the parameters used by most well-known papers in this literature (e.g., Bansal and Yaron 2004), in the example above the agent would strictly prefer option B. For HS preferences, we show that Stochastic Impatience always fails as long as the range of utilities of consumption is large enough (e.g., when the utility function is unbounded either above or below). For example, using this model with the parameters of Tallarini Jr (2000), again option B would be chosen in the example above. More generally, Stochastic Impatience is violated for all common parametrizations of the two leading models that separate time and risk preferences.

We then establish a more general result, going beyond EZ and HS. To this end, instead of considering the full space of dynamic preferences over temporal lotteries, as in EZ, we show that it suffices to look at their static implications – that is, how they evaluate lotteries over future consumption streams at a given point in time. For example, in EZ or HS we can obtain such preferences by focusing on their restrictions to lotteries over streams starting in any fixed period and in which all uncertainty is resolved in some given future period. This restriction contains all the information needed for our results regarding the separation between time and risk preferences. Furthermore, as we will discuss in Section 4, it allows us to use not only a simpler framework, but also one that applies to a large class of models, as any dynamic model has implications when restricted to this subdomain of lotteries, independently of how they are defined in richer dynamic settings (e.g., whether they are dynamically consistent or not).

We assume that (i) without risk, the preference relation coincides with discounted utility, so that it admits a representation $\sum D(t)u(x(t))$ for a decreasing D and an increasing u; and that (ii) for choices over lotteries over streams, the preference relation satisfies the Expected Utility postulates. These assumptions hold in the restrictions of most models to static settings, including EZ and HS. Note that little is assumed about the discount function D.

We show that, in the space that we consider, any preference relation that satisfies these two assumptions admits a representation of the form $\mathbb{E}\big[\phi\big(\sum D(t)u(x(t))\big)\big]$. The aggregator applied to discounted utilities ϕ introduces an additional curvature that permits a separation between attitudes towards time and risk. This is a well-known functional form, and following the literature we refer to it as a Kihlstrom-Mirman (KM) representation, as it can be seen as an application of the multi-attribute function of Kihlstrom and Mirman (1974) to the context of time. While this functional form has known drawbacks when used dynamically, it will help us illustrate our results

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since any model whose static implications satisfy assumptions (i) and (ii) admits such a representation in our setting.

Next, we use a behavioral definition of *Residual Risk Aversion*: the amount of risk aversion beyond what is captured by EIS. In a KM representation, Residual Risk Aversion is captured by the curvature of ϕ: a concave (resp. convex) ϕ is equivalent to Residual Risk Averse (resp. Seeking) behavior.

Our general result then shows that Stochastic Impatience imposes an upper bound on Residual Risk Aversion. In the language of the KM representation, we show that one can always construct a violation of Stochastic Impatience if ϕ is *more* concave than the log function. Conversely, Stochastic Impatience holds if ϕ is *less* concave than the log. This result unifies the findings from EZ and HS, showing that the bound on Residual Risk Aversion holds more generally, in any model that satisfies our two basic assumptions.[2]

Overall, there are two implications of our results. If Stochastic Impatience is taken as an appealing property – normatively or behaviorally – our results highlight an issue with the modeling of the separation of time and risk preferences, including all common parametrizations of all leading approaches. In Section 5 we discuss possible ways to maintain Stochastic Impatience without sacrificing the fit of empirical data.

If, instead, Stochastic Impatience is viewed as a behavioral property that may or may not hold, our results provide an easy test of whether risk aversion is significantly above the inverse of EIS: this can be established by documenting a single violation of Stochastic Impatience. It is a much simpler test than those used in the literature, where the two parameters are estimated indirectly using multiple questions and assuming specific functional forms.

This is not the first paper to point out potentially problematic implications of how separation between time and risk is typically modeled. Epstein et al. (2014) note that common parameterizations of EZ imply an unrealistically strong preference for early resolution of uncertainty. We show that with the same parameters, we also have a violation of Stochastic Impatience – a property that is distinct from preference for early or late resolution of uncertainty. Bommier et al. (2017) show that many known models that separate time and risk preferences, including common specifications of EZ, violate a property of Monotonicity that may be seen as normatively appealing. Stochastic Impatience is distinct from their Monotonicity: for example, EZ with both risk aversion and the inverse of EIS less than 1 satisfies Stochastic Impatience but not Monotonicity; conversely, HS always satisfies Monotonicity but violates Stochastic Impatience when the utility range of prizes is large enough. Lastly, a companion paper, DeJarnette et al. (2018), studies Risk Aversion over Time Lotteries and shows how that property is incompatible with Stochastic Impatience within a broad class of models, although in a different formal setup.

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[2]In Appendix A we provide two extensions. First, to continuous time, showing that equivalent results hold. Second, to the case of non-Expected Utility, where we show that exhibiting First Order Risk Aversion implies violations of Stochastic Impatience.

2 Framework

We study a preference relation on lotteries over consumption streams. Consider an interval of per-period consumption $C \subset \mathbb{R}_+$ and a set of dates $T = \{1, \ldots, \bar{t}\}$, where $\bar{t}$ is either finite or infinite.[3] A consumption program $\mathbf{x} = (x(1), x(2), \ldots, x(\bar{t}))$ yields consumption $x(t) \in C$ in period $t \in T$. Let $\mathcal{X} = C^T$ be the set of consumption programs and let Δ be the set of all simple probability measures over it. Let $\succcurlyeq$ be a complete and transitive preference relation over Δ.

We abuse notation and refer to $\mathbf{x} \in \mathcal{X}$ both as the consumption program and as the lottery that gives this consumption program with certainty (i.e., the Dirac measure on $\mathbf{x}$). To further simplify notation, we denote by $(c, t, x) \in C \times T \times C$ the stream that gives c in every period until $t - 1$ and x from t onwards:

$$(c, c, \ldots, \underbrace{c}_{t-1}, \underbrace{x}_{t}, x, \ldots, \underbrace{x}_{\bar{t}}).$$

We consider the static space of lotteries over streams, and not the more complex space of temporal lotteries used in Kreps and Porteus (1978) or EZ, because this is sufficient for our purposes and allows for a much simpler treatment that does not sacrifice generality. To see why, note that any model over temporal lotteries induces preferences over this subdomain.[4] Crucially, these static preferences will contain all the information pertaining to the separation of time and risk preferences that is relevant for our analysis. In addition, restricting attention to this subdomain allows us not only to conduct our analysis in a much simpler setup, but also to derive results for a richer class of models, independently of how they are defined dynamically.

We now introduce the main property studied in this paper:

Definition 1 (Stochastic Impatience)**.** The relation $\succcurlyeq$ satisfies Stochastic Impatience if for any $t_1, t_2 \in T$ with $t_1 < t_2$, and any $c, x_1, x_2 \in C$ with $x_1 > x_2 > c$,

$$\frac{1}{2}(c, t_1, x_1) + \frac{1}{2}(c, t_2, x_2) \succcurlyeq \frac{1}{2}(c, t_2, x_1) + \frac{1}{2}(c, t_1, x_2). \tag{1}$$

Stochastic Impatience states that the individual prefers the lottery in which she either starts receiving higher payments earlier or lower payments later. It can be

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[3]We focus on real-valued consumption to simplify exposition and notation and on discrete time to facilitate the connection with applications. It is immediate to extend our results to arbitrary consumption spaces. Appendix A presents the extension to continuous time, where analogous results hold. We start from date one (and not zero) to allow the introduction of additional consumption at time zero that is not subject to uncertainty. As discussed below, this feature allows us to incorporate some commonly used models, such as EZ.

[4]Formally, lotteries over streams are embedded within temporal lotteries as lotteries starting from any fixed period with the timing of resolution of uncertainty held constant at any other future period. For example, starting from preferences over temporal lotteries and holding the time-zero consumption fixed and assuming that uncertainty is resolved between periods zero and one, one obtains preferences over lotteries over streams. Thus, each model identifies a preference relation over Δ.

seen as a stochastic counterpart of the standard impatience property – the individual prefers higher payments sooner – to a stochastic environment. As mentioned in the introduction, a related argument for its appeal can be made by decomposing each alternative into two parts. Note that both options offer the same basic lottery that pays x_2 starting at either t_1 or t_2 along with an increment of $x_1 - x_2$. The difference between the two options is when the increment is paid: for the option on the left, it is paired with the earlier date t_1, while the option on the right pairs it with the later date t_2. Insofar as the agent prefers to obtain it sooner, the option on the left may be preferred.

To our knowledge, Stochastic Impatience is a new property – with the exception that a version of it appears, in a different setting, in a companion paper (DeJarnette et al., 2018).[5] We say that Stochastic Impatience fails if the condition in Definition 1 does not hold, i.e., if there exist $t_1 < t_2$ and $x_1 > x_2 > c$ such that (1) fails.

In Expected Discounted Utility (henceforth EDU), impatience – a preferences for earlier rewards – and Stochastic Impatience are equivalent:

Observation 1 (EDU satisfies Stochastic Impatience). Consider a preference relation $\succcurlyeq$ that admits a representation $\mathbb{E}\left[\sum_T D(t)u(x(t))\right]$ for some strictly increasing u. Then, $\succcurlyeq$ satisfies Stochastic Impatience if and only if D is weakly decreasing.

We conclude by noting that different versions of Stochastic Impatience may also be considered. For example, the increase in consumption may last a fixed number of periods, even only one, instead of being permanent — a condition that is easier to test. In Appendix A we show that, under general conditions, this is equivalent to the version above.[6]

3   Stochastic Impatience in EZ and HS

3.1   Epstein-Zin preferences

We begin our formal analysis by considering the most widely used model that separates time and risk preferences: the standard CRRA-CES version of EZ. Let the consumption space be $C = \mathbb{R}_{++}$ and let $\bar{t} = +\infty$. This model admits the following recursive representation:

$$V_t = \left\{ (1 - \beta)\, x(t)^{1 - \frac{1}{\psi}} + \beta \left[E_t \left(V_{t+1}^{1-\alpha} \right) \right]^{\frac{1 - \frac{1}{\psi}}{1 - \alpha}} \right\}^{\frac{1}{1 - \frac{1}{\psi}}} \tag{2}$$

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[5]That paper considers a different formal setup of prize-date pairs, instead of streams. It thus considers a version of stochastic impatience in which the payment is given at one specific period instead of constituting a constant change in the stream.

[6]Specifically, in Appendix A we show that under the conditions of Proposition 4 below, these properties are equivalent in continuous time (and in discrete time with arbitrary small time periods). In discrete time, they are not equivalent for issues pertaining to the discreteness: however, qualitative conclusions remain identical.

where $\alpha \in \mathbb{R}_+\backslash\{1\}$ is the coefficient of relative risk aversion and $\psi \in \mathbb{R}_+\backslash\{1\}$ is the EIS. When $\alpha = \frac{1}{\psi}$, the model coincides with EDU.

The following result fully characterizes Stochastic Impatience in this model (all proofs appear in Appendix C):

Proposition 1. *Let $\succcurlyeq$ be a preference relation that admits a representation as in (2). Then, it satisfies Stochastic Impatience if and only if either (i) $\alpha \leq \frac{1}{\psi}$, or (ii) $\alpha < 1$.*

Stochastic Impatience holds in the CRRA-CES version of EZ either when the coefficient of risk aversion is below one or when risk aversion is less than the inverse of EIS. Conversely, Stochastic Impatience is violated – in the sense that there are instances in which the ordering in (1) fails – when risk aversion α is above the inverse of EIS, $\frac{1}{\psi}$, as well as above one.

To understand the intuition for this result, consider again the decision problem given in the introduction: a choice between a lottery that gives either a 20% increase starting today or a 10% increase starting next year and a lottery that gives either a 10% increase starting today or a 20% increase starting next year. Of the four possible outcomes, the best is 20% starting today, the worst is 10% next year, while the other two are intermediate. The first lottery therefore involves the best and the worst outcomes, while the second one features the two 'intermediate' ones. Thus, the first lottery has more spread in discounted utility but also has a higher expected discounted utility – since the higher discounting is applied to the smaller amount. When $\alpha = \frac{1}{\psi}$, i.e., with EDU, the agent cares only about the expected discounted utility, and thus strictly prefers the first option. But when risk aversion is increased fixing EIS, the individual starts disliking the spread in discounted utilities – which favors the second option. When risk aversion is high enough, this second effect prevails, leading the individual to prefer the second option and violate Stochastic Impatience. Proposition 1 provides the exact condition for when this is the case: $\alpha > \max\{\frac{1}{\psi}, 1\}$.

The result above should be understood in light of the parameters used in the wide literature that adopts EZ. All applications that we are aware of assume $\alpha > \max\{\frac{1}{\psi}, 1\}$. Indeed, the possibility of incorporating a risk aversion greater than the inverse of EIS is a primary reason for adopting this model; and assuming a relative risk aversion above one is also typically assumed to fit finance data. For example, Bansal et al. (2016) study annual data from 1930 to 2015 and estimate a coefficient of risk aversion of $\alpha = 9.67$ and EIS of $\psi = 2.18$ (see Example 1 below for other references). Proposition 1 shows that Stochastic Impatience fails in this range.

A different strand of the literature (that typically does not adopt EZ) has instead argued for an EIS less than one.[7] Still, incorporating this restriction to EZ implies that one cannot allow for a coefficient of risk aversion α greater than $\frac{1}{\psi}$ while avoiding violations of Stochastic Impatience.

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[7] A discussion of this vast empirical and theoretical debate is outside the scope of this paper. See the discussions in Campbell (1999), Attanasio and Weber (2010), Campbell (2003) and, for more recent contributions, Gruber (2013), Ortu et al. (2013), Crump et al. (2015), and Best et al. (2017).

We now provide an example of violations of Stochastic Impatience.

Example 1. Recall lotteries A and B described in the introduction:

A. With equal probability, permanently increase consumption by either 20% starting today, or by 10% starting next year;

B. With equal probability, permanently increase consumption by either 10% starting today, or by 20% starting next year.

Stochastic Impatience implies that A is preferred to B. However, B is preferred adopting the EZ model with the parameters of many known papers: Bansal and Yaron (2004) ($\alpha = 10$, $\beta = 0.998$, $\psi = 1.5$), Bansal et al. (2016) ($\alpha = 9.67$, $\beta = .999$, $\psi = 2.18$), Nakamura et al. (2017) ($\alpha = 9$, $\beta = 0.99$, $\psi = 1.5$), and Colacito et al. (2018) ($\alpha = 10$, $\beta = 0.97$, $\psi = 1.1$).[8]

Within EZ, Stochastic Impatience puts restrictions on the attitude towards the timing of resolution of uncertainty. The agent prefers early (late) resolution of uncertainty whenever α is higher (smaller) than $\frac{1}{\psi}$ (Epstein and Zin, 1989). Proposition 1 shows that when $\alpha > 1$, preference for early resolution of uncertainty coincides with violations of Stochastic Impatience; or, put differently, Stochastic Impatience implies a weak preference for *late* resolution of uncertainty. This observation links our results to those in Epstein et al. (2014), who argue that the parameters used in much of the literature imply levels of preference for early resolution of uncertainty that may be considered implausible. Here we show that these same parameters imply a violation of Stochastic Impatience. We emphasize that this implication is derived even though Stochastic Impatience is conceptually independent of preferences over the timing of resolution of uncertainty. The reason is that in the context of EZ, the two notions solely depend on the parameters α and ψ.

To summarize, we have seen that with the parameters commonly used to fit macroeconomic and financial data, the CRRA-CES specification of EZ is bound to violate Stochastic Impatience.

3.2 Risk-Sensitive Preferences

We now show that an analogous result holds for the Risk Sensitive preferences of HS (see also Strzalecki 2013 and Bommier et al. 2017). Let $\bar{t} = +\infty$. HS preferences

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[8]Even if one takes EIS< 1, option B is still preferred to A if risk aversion is high enough. For example, with $\alpha = 10$ and $\beta = 0.998$, option B is preferred for any $\psi > 0.2576$. With less risk aversion, violations of Stochastic Impatience require higher prizes. For example, with the parameters of Nakamura et al. (2013) ($\alpha = 6.4$, $\beta = 0.967$, $\psi = 2$), a violation is observed with a low prize of 20% and a high prize of 30% of per-period consumption. With the parameters of Barro (2009) ($\alpha = 4$, $\beta = 0.948$, and $\psi = 2$), it is observed with a low prize of 35% and a high prize of 40%. With even less risk aversion, closer to one, violations of Stochastic Impatience require higher and higher prizes: with $\alpha = 2$, $\beta = 0.99$, and $\psi = 2$, one needs a low prize of 120% and a high prize of 140%. Reducing α to 1.2, we obtain a violation of Stochastic Impatience with prizes of 1100% and 1200%.

admit the recursive representation:

$$V_t = u(x(t)) - \beta \cdot \frac{1}{k} \cdot \ln \left(E \left[e^{-kV_{t+1}} \right] \right).$$ (3)

Proposition 2. *Suppose $\succcurlyeq$ admits a representation as in (3). Stochastic impatience holds if and only if $\sup_{x \in C}\{u(x)\} - \inf_{x \in C}\{u(x)\} \leq -\frac{\ln(\beta)}{k\beta(1-\beta)}$.*

The result above shows that Stochastic Impatience is violated if the utility range of prizes is large enough. This is necessarily the case if the utility function $u : C \rightarrow \mathbb{R}$ is unbounded above or below (such as with a CARA utility function and an unbounded consumption space). Otherwise, Stochastic Impatience requires the amount of additional risk aversion (k) and the maximum gap in per-period utility $(\sup u(x) - \inf u(x))$ to be small enough.

Like in the case of EZ, Proposition 2 only shows that violations can be constructed, but does not specify which ones. We now give an example of such violation using an influential parameterization:

Example 2. Tallarini Jr (2000) considers the specification in (3) with $C = \mathbb{R}_{++}$, $u(x) = \ln(x)$, and $k = (1-\beta)(\xi-1)$, where ξ is the coefficient of relative risk aversion with respect to atemporal wealth gambles. First note that, since u is unbounded, Stochastic Impatience fails for any parameter values. Tallarini Jr (2000) shows that the model is able to match some key moments in asset pricing for some $(\xi, \beta) \in [46, 180] \times [.991, .999]$. Consider again options A and B described in Example 1, where Stochastic Impatience posits that option A is preferred. Within HS, adopting any of the parameters of Tallarini Jr (2000), option B is preferred instead.[9]

4 A general result

The previous results show how stochastic impatience imposes a bound of the separation of time and risk preferences in two well-known models in the literature. In this section we explore the extent to which similar results hold more generally. To avoid continuity issues, in the remainder we assume that the space of per-period consumption is a compact interval: $C = [\underline{x}, \overline{x}] \subset \mathbb{R}_+$. We focus on preferences that satisfy the following two assumptions.

Assumption 1 (Discounted Utility without risk)**.** *There exist a strictly increasing and continuous function $u : [\underline{x}, \overline{x}] \rightarrow \mathbb{R}_+$ and a strictly decreasing function $D : T \rightarrow [0, 1]$ such that for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$*

$$\boldsymbol{x} \succcurlyeq \boldsymbol{y} \quad \Leftrightarrow \quad \sum_{t \in T} D(t)u(x(t)) \geq \sum_{t \in T} D(t)u(y(t)).$$

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[9]Specifically, whenever both prizes exceed a proportion $\beta^{\frac{1}{(1-\xi)(1-\beta)}} - 1$ of background consumption, we have a violation of Stochastic Impatience. For example, with $\beta = .998$ and $\xi > 11.5$, we have $\beta^{\frac{1}{(1-\xi)(1-\beta)}} - 1 < 10\%$, so option B is preferred to option A.

Assumption 2 (Expected Utility). *The following hold:*

1. *For all $p, q, r \in \Delta$ and $\lambda \in (0, 1)$,*

$$p \succcurlyeq q \Leftrightarrow \lambda p + (1 - \lambda)r \succcurlyeq \lambda q + (1 - \lambda)r;$$

2. *For all $p, q, r \in \Delta$ with $p \succ q \succ r$, there exist $\alpha, \beta \in (0, 1)$ such that $\alpha p + (1 - \alpha)r \succ q \succ \beta p + (1 - \beta)r$.*

Assumption 1 posits that in the absence of risk, preferences can be modeled using Discounted Utility with generic discount function D. This is true for the vast majority of models used to study time and risk preferences, including EZ and HS, and allows for many types of discounting (e.g., exponential, hyperbolic, and quasi-hyperbolic). Assumption 2 posits the postulates of Expected Utility under risk. Again, these requirements are satisfied by most models in the literature, including EZ and HS. In Appendix B we show that similar results to those presented below hold even allowing for non-Expected Utility.

For the purposes of our analysis, it will be useful to note that Assumptions 1 and 2 yield the following representation:

Lemma 1. $\succcurlyeq$ *satisfies Assumptions 1 and 2 if and only if there exist a strictly increasing and continuous $u : [\underline{x}, \overline{x}] \to \mathbb{R}_+$ with $u(\underline{x}) = 0$, a strictly decreasing $D : T \to [0, 1]$, and a strictly increasing $\phi : [0, u(\overline{x})] \to \mathbb{R}$ such that $\succcurlyeq$ is represented by*

$$V(p) = \mathbb{E}_p \Big[\phi \Big(\frac{1}{\sum_T D(t)} \sum_T D(t)u(x(t)) \Big) \Big].$$

Conditional on u and D, ϕ is unique up to a positive affine transformation.

We call this representation a Kihlstrom-Mirman (KM) representation, as it can be seen as an application of the multi-attribute function of Kihlstrom and Mirman (1974) to the context of time.[10] Fixing D, the curvature of u captures EIS in the KM model, whereas risk aversion is captured by the curvature of $\phi \circ u$, so that ϕ is an additional curvature used only in the case of risk. Therefore, the KM representation will be useful to discuss the separation between time and risk preferences in Δ.

While the model itself is known, Lemma 1 shows that it is axiomatically characterized by Assumptions 1 and 2. Despite being in itself a very simple result that follows from standard arguments,[11] this representation provides a convenient functional form

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[10] This application to time preferences is discussed, for example, in EZ. The functional form has been derived, in a different setup, by DeJarnette et al. (2018). A similar functional form was used, but not derived, by Andersen et al. (2017), to study intertemporal utility and correlation aversion, by Abdellaoui et al. (2017), to study different questions on time and risk, as well as by Edmans and Gabaix (2011) and Garrett and Pavan (2011) in applied contexts.

[11] Assumption 1 guarantees a Discounted Utility representation without risk; Assumption 2 guarantees an Expected Utility representation with a given Bernoulli utility V. Since V and the Discounted Utility representation must be ordinally equivalent, there exists a strictly increasing function ϕ that makes them equal.

to study the static implications of most models used in the literature, including EZ and HS, as their static implications satisfy both of our assumptions. Importantly, this representation contains all the information on the separation of time and risk preferences – in the function ϕ – and thus provides a useful new way of looking at these models. Interestingly, their KM representation takes familiar shapes: below, we show that Expected Discounted Utility, EZ, and HS correspond to linear, CRRA, and CARA functions ϕ, respectively.

Example 3 (Expected Discounted Utility)**.** If ϕ is affine, $\succcurlyeq$ can be represented by $\mathbb{E}\left[\sum_T D(t)u(x(t))\right]$, the standard case of EDU.

Example 4 (EZ with CRRA-CES)**.** Fix a consumption for time zero at $c \in [\underline{x}, \overline{x}]$, and consider a preference $\succcurlyeq$ over Δ represented by (2). Then, $\succcurlyeq$ also admits a KM representation, with $u(x) = \frac{x^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}$, $D(t) = \beta^t$, and

$$
\phi\left(z\right) = \begin{cases} z^{\frac{1-\alpha}{1-\frac{1}{\psi}}} & \text{if } \alpha < 1 < \psi \\ -\left(-z\right)^{\frac{1-\alpha}{1-\frac{1}{\psi}}} & \text{if } \alpha > 1 > \psi \\ -z^{\frac{1-\alpha}{1-\frac{1}{\psi}}} & \text{if } 1 < \alpha,\ 1 < \psi \\ \left(-z\right)^{\frac{1-\alpha}{1-\frac{1}{\psi}}} & \text{if } \alpha < 1,\ \psi < 1 \end{cases} .
\tag{4}
$$

Example 5 (HS)**.** Fix a consumption for time zero at $c \in [\underline{x}, \overline{x}]$, and consider a preference relation $\succcurlyeq$ over Δ represented by (3). Then, $\succcurlyeq$ also admits a KM representation with $D(t) = \beta^t$ and $\phi(x) = -\exp\left(-kx\right)$.

Before proceeding, we emphasize that Examples 4 and 5 do *not* imply that the KM model includes EZ and HS as special cases. Rather, they imply that the static implications of EZ and HS on Δ must admit a KM representation. On the one hand, if applied dynamically on the larger space of temporal lotteries with different current-period consumption, not only the models are not nested, but the KM model – applied dynamically without modifications – in general leads to dynamically inconsistent behavior (see EZ for a careful discussion). By contrast, both EZ and HS are recursive and dynamically consistent. These potential limitations of the KM functional in dynamic context are inconsequential in our static setup.

On the other hand, our results show that one can view EZ or HS as being composed of a *collection* of KM representations, where ϕ varies with the timing of resolution of uncertainty and with current consumption in order to allow for recursivity and dynamic consistency (where, importantly, the utility of the current consumption, which is deterministic, is added *outside* of the aggregator ϕ).

4.1   Residual Risk Aversion

We now introduce a behavioral notion that we call *Residual Risk Aversion*, which aims to capture the individual's additional risk aversion relative to the curvature

already implied by EIS. This is an adaptation to our framework of Traeger (2014)'s notion of Intertemporal Risk Aversion. [12]

Definition 2. Let $\succcurlyeq$ be a preference relation over Δ. We say that $\succcurlyeq$ displays Residual Risk Aversion if for any $a, b, c, d, x \in [\underline{x}, \overline{x}]$ such that

$$(a, d, x, x, \dots) \sim (b, b, x, x, \dots) \ \text{ and } \ (d, a, x, x, \dots) \sim (c, c, x, x, \dots)$$

we have

$$\frac{1}{2}(b, b, \dots) + \frac{1}{2}(c, c, \dots) \succcurlyeq \frac{1}{2}(a, a, \dots) + \frac{1}{2}(d, d, \dots).$$

We say that $\succcurlyeq$ displays Residual Risk Seeking/Neutrality if the above is instead $\preccurlyeq/\sim$.

To understand this definition, suppose $\succcurlyeq$ satisfies Assumption 1 with utility u and discount function D. If $(a, d, x, x, \dots) \sim (b, b, x, x, \dots)$ and $(d, a, x, x, \dots) \sim (c, c, x, x, \dots)$ it must be the case that

$$D(1)u(a) + D(2)u(d) = [D(1) + D(2)]u(b) \text{ and } D(1)u(d) + D(2)u(a) = [D(1) + D(2)]u(c).$$

Thus, $u(a) + u(d) = u(b) + u(c)$. Notice also that we must have either $a > b > c > d$ or $a < b < c < d$, and that these depend only on how $\succcurlyeq$ ranks streams without risk. Now suppose that we take a lottery that returns with equal chances the constant streams a or d; and a lottery that returns with equal chances the constant streams b or c. If all risk aversion is included in the curvature of u, these two lotteries must be indifferent to one other, as $u(a) + u(d) = u(b) + u(c)$. But if the individual displays additional risk aversion not captured by the curvature of u, then either $a > b > c > d$ or $a < b < c < d$ implies that she should prefer the lottery between the constant streams b and c, in which the utility spread is smaller.

We first link Residual Risk Aversion to the properties of the KM representation, showing that the curvature of ϕ is related to Residual Risk Aversion in a similar way to how the curvature of the Bernoulli utility function is related to risk aversion in standard Expected Utility theory:

Proposition 3 (Residual Risk Attitudes and the curvature of ϕ). *Suppose $\succcurlyeq$ admits a KM representation (ϕ, D, u). Then, $\succcurlyeq$ displays Residual Risk Aversion if and only if ϕ is concave. Moreover, $\succcurlyeq$ displays Residual Risk Seeking/Neutrality if and only if ϕ is convex/affine.*

It follows from Proposition 3 that Residual Risk Neutrality characterizes EDU given Assumptions 1 and 2, and that EZ allows one to introduce Residual Risk Aversion:

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[12]Similar to Proposition 3 and Observation 4 below, Traeger also gave a functional characterization of attitudes towards intertemporal risk aversion and a related comparative notion in his framework.

Observation 2 (EDU is characterized by Residual Risk Neutrality)**.** Suppose $\succcurlyeq$ satisfies Assumptions 1 and 2. Then, it admits an EDU representation if and only if it displays Residual Risk Neutrality.

Observation 3 (EZ preferences)**.** Suppose $\succcurlyeq$ admits a representation as in (2). Then $\succcurlyeq$ displays Residual Risk Aversion/Neutrality/Seeking if and only if $\alpha \geq / = / \leq \frac{1}{\psi}$.

Recall that the CRRA-CES version of EZ displays a preference for early (a preference for late/neutrality towards) resolution of uncertainty if $\alpha > (< / =)\frac{1}{\psi}$. Therefore, in this model, $\succcurlyeq$ displays Residual Risk Aversion (Seeking/Neutrality) if and only if, in the space of temporal lotteries, there is a preference for early (a preference for late/neutrality towards) resolution of uncertainty.

We can also introduce comparative notions.

Definition 3. Consider two preference relations $\succcurlyeq_1$ and $\succcurlyeq_2$ over Δ. We say that $\succcurlyeq_1$ has more Residual Risk Aversion than $\succcurlyeq_2$ if they coincide on degenerate lotteries and if, for all $a > b > c > d$,

$$\frac{1}{2}(b, b, \dots) + \frac{1}{2}(c, c, \dots) \succcurlyeq_2 \frac{1}{2}(a, a, \dots) + \frac{1}{2}(d, d, \dots)$$

implies

$$\frac{1}{2}(b, b, \dots) + \frac{1}{2}(c, c, \dots) \succcurlyeq_1 \frac{1}{2}(a, a, \dots) + \frac{1}{2}(d, d, \dots).$$

The comparative notion above parallels standard ones for risk and ambiguity aversion (Ghirardato and Marinacci 2002): it only compares preferences that have the same ranking on risk-free streams; for any two such preferences, it defines one as more Residual Risk Averse than the other if it prefers the smallest "utility spread" at least as much as the other.

Since, as shown in Proposition 3, the curvature of ϕ determines Residual Risk Aversion in KM representations, we can obtain a comparative notion analogous to risk aversion in standard Expected Utility theory:

Observation 4. Let $\succcurlyeq_1$ and $\succcurlyeq_2$ be two preferences with KM representations (ϕ_1, u, D) and (ϕ_2, u, D), respectively. Then, $\succcurlyeq_1$ has more Residual Risk Aversion than $\succcurlyeq_2$ if and only if there exist a strictly increasing and concave function $f : \mathbb{R} \to \mathbb{R}$ such that $\phi_1 = f \circ \phi_2$.

4.2 Stochastic Impatience limits Residual Risk Aversion

We are now ready to state our main general result: Stochastic Impatience imposes bounds on Residual Risk Aversion. In the proposition below, we say that a function ϕ is more concave/convex than the log if $\phi = f \circ \ln$ for some concave/convex f.

Proposition 4. *Let $C = [\underline{x}, \overline{x}]$. Let $\succcurlyeq$ be a preference relation over Δ that satisfies Assumptions 1 and 2 and let (ϕ, u, D) be a KM representation of $\succcurlyeq$.*

(i) If ϕ is weakly less concave than the log, then $\succcurlyeq$ satisfies Stochastic Impatience;

(ii) If ϕ is strictly more concave than the log, then $\succcurlyeq$ violates Stochastic Impatience;

(iii) There exists some preference relation $\succcurlyeq'$ such that $\succcurlyeq'$ has more Residual Risk Aversion than $\succcurlyeq$ and violates Stochastic Impatience;

(iv) Let $\succcurlyeq'$ be another preference relation over Δ that satisfies Assumptions 1 and 2 and has more Residual Risk Aversion than $\succcurlyeq$. Then:

> *(a)* if $\succcurlyeq$ violates Stochastic Impatience, so does $\succcurlyeq'$;
> *(b)* if $\succcurlyeq'$ satisfies Stochastic Impatience, so does $\succcurlyeq$.

All these results have the same interpretation: no matter what are the preferences without risk – that is, no matter what EIS is –, adding enough Residual Risk Aversion *must* lead to a violation of Stochastic Impatience. In turn, this means that Stochastic Impatience imposes a bound on Residual Risk Aversion. One way to identify this bound is through the curvature of ϕ in the KM representation. This is the content of parts *(i)* and *(ii)*: whenever ϕ is more concave than the log, Stochastic Impatience must be violated; and less concavity than the log means that Stochastic Impatience holds. Alternatively, the result can be stated in terms of comparative Residual Risk Aversion, the content of parts *(iii)* and *(iv)*: if we increase Residual Risk Aversion, Stochastic Impatience must eventually be violated; and comparative notions or Residual Risk Aversion are fully reflected in whether Stochastic Impatience holds.

The proof of Proposition 4 follows the same intuition outlined in Section 3.1 for the special case of EZ: Stochastic Impatience implies that the agent should prefer the lottery with a higher average but also higher spread in discounted utilities; under EDU such spread does not matter, but Residual Risk Aversion induces the agent to be averse to it.

We note that parts *(i)* and *(ii)* of Proposition 4 together do not constitute an if and only if statement: when ϕ is neither more concave nor more convex than the log, the proposition is silent as to whether Stochastic Impatience holds. This is due to the discreteness of time intervals: in Appendix A we consider the model with continuous time and show that, in that case, Stochastic Impatience holds if and only if ϕ is less concave than the log.

5 Discussion

In this paper we show that under discounted and Expected Utility (Assumptions 1 and 2), Stochastic Impatience puts an upper bound on Residual Risk Aversion. This bound rules out the parameters used in most applications of EZ and HS. Thus, to

maintain Stochastic Impatience, one must either relax these assumptions or lower Residual Risk Aversion.

In terms of relaxing our assumptions, in Appendix B we extend our analysis beyond Expected Utility, by considering a very broad class of models which includes probability weighting (Rank-Dependent Utility, Quiggin 1982, and Cumulative Prospect Theory, Tversky and Kahneman 1992) and Disappointment Aversion (Gul, 1991). Indeed, many applications of such models to our context have already been suggested in the literature, starting in the original paper of Epstein and Zin (1989).[13] We show that models in this class always violate Stochastic Impatience whenever they exhibit First Order Risk Aversion – as is the case in almost all specifications that use them.[14] One may instead want to relax our other assumption of discounted utility without risk. This is done by dropping additive separability, such as in models with habit formation or memorable consumption. Qualitatively, our results extend beyond separable preferences, in the sense that for any such model there is still a bound on how high risk aversion can be for a fixed EIS.[15] However, since there are many forms of non-separable preferences, a unified result with a clear-cut bound on risk aversion for a fixed EIS cannot be obtained without additional assumptions.

An alternative route is instead to incorporate other features into the model that do not require high Residual Risk Aversion. Large coefficients of risk aversion – far above 1 and in many cases as high as 10 or much more – are often needed because in typical models an investor's unwillingness to take financial or similar risks is solely due to risk aversion. In practice, investors may also be affected by other aspects, such as ambiguity aversion/robustness concerns, pessimism, or rational inattention. If these aspects are relevant but omitted from the model, risk aversion may be overestimated, possibly to unrealistic parameters. For example, if some of the equity premium is due to ambiguity aversion, incorporating it into the model may allow for much lower coefficients of risk aversion (Barillas et al., 2009). This would reduce the preference for early resolution of uncertainty to more realistic levels (Epstein et al., 2014) and, more to the point, allow for Stochastic Impatience to hold: since the latter is based on objective lotteries, it is unaffected by ambiguity aversion. In general, any feature that reduces the individual's willingness to undertake financial risk without modifying her attitude towards objective lotteries, as discussed in the surveys of Backus et al. (2004), Epstein and Schneider (2010), and Hansen and Sargent (2014), could provide

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[13]See Backus et al. (2004) and references therein.

[14]This includes, for example, pessimistic weighting functions (underweighting the probabilities of good outcomes) in Rank-Dependent Utility, or a positive coefficient of disappointment aversion in Gul (1991). See Segal and Spivak (1990) for a definition and general discussion of First Order Risk Aversion. The key intuition is that First Order Risk Aversion implies extreme risk aversion with respect to discounted utilities in a small neighborhood of certainty, which, by the same intuition as above, leads to violations of Stochastic Impatience.

[15]Intuitively, even if we weaken Assumption 1 while maintaining Assumption 2, a concave enough ϕ makes the value of any lottery be arbitrarily close to the value of its worst outcome, thus generating a violation of Stochastic Impatience.

a way to reconcile the empirical fit of the model with Stochastic Impatience. Thus, one possible message of our paper is that it highlights the importance of including other relevant behavioral aspects instead of simply increasing risk aversion fixing EIS — because this may have unappealing implications.

Appendix

A Continuous Time and Additional Results

In this appendix, we consider a continuous time formulation of the model from Section 4. As with discrete time, let $C = [\underline{x}, \overline{x}] \subset \mathbb{R}_+$ denote the space of per-period consumption. The set of dates is now $T = [0, \overline{t}]$, where $\overline{t} > 0$ may be $+\infty$.

For each $\delta \in \mathbb{R}_{++} \cup \{+\infty\}$, let (c, t, x, δ) denote the consumption stream that returns $c \in C$ for each time $\tilde{t} \notin [t, t+\delta)$ and returns $x \in C$ for $\tilde{t} \in [t, t+\delta) \cap T$. Note that $(c, t, x, +\infty)$ denotes the consumption stream that returns c until date t and x from t onwards.

We consider preferences $\succcurlyeq$ that can be represented by

$$V(p) = \mathbb{E}_p\Big[\phi\Big(\frac{\int_0^{\overline{t}} D(t)u(x(t))dt}{\int_0^{\overline{t}} D(t)dt}\Big)\Big],$$

for lotteries such that the integrals are well defined, where $u : C \to \mathbb{R}_+$ is continuous, strictly increasing, and satisfies $u(\underline{x}) = 0$, $D : T \to [0, 1]$ is continuous and strictly decreasing, and $\phi : [0, u(\overline{x})] \to \mathbb{R}$ is strictly increasing.

Recall that Stochastic Impatience states that it is preferred to associate higher prizes to earlier dates. The analogous notion of Stochastic Impatience to continuous time is as follows:

Definition 4 (Stochastic Impatience'). $\succcurlyeq$ satisfies Stochastic Impatience' if for any $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$, and any $c, x_1, x_2 \in C$ with $x_1 > x_2 > c$

$$\frac{1}{2}(c, t_1, x_1, +\infty) + \frac{1}{2}(c, t_2, x_2, +\infty) \succcurlyeq \frac{1}{2}(c, t_2, x_1, +\infty) + \frac{1}{2}(c, t_1, x_2, +\infty). \quad (5)$$

Note that in the definition above, any increase in consumption is always permanent. In the example from the introduction, the options involved either a permanent increase of 20% starting at some future date or a permanent increase of 10% starting at some future date. What would happen if, instead, we also allowed the increases to be paid only for a finite period of time? Consider the following stronger version.

Definition 5 (Strong Stochastic Impatience). $\succcurlyeq$ satisfies Strong Stochastic Impatience if for any $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$, any $\delta \in \mathbb{R}_{++} \cup \{+\infty\}$ and any $c, x_1, x_2 \in C$ with $x_1 > x_2 > c$

$$\frac{1}{2}(c, t_1, x_1, \delta) + \frac{1}{2}(c, t_2, x_2, \delta) \succcurlyeq \frac{1}{2}(c, t_2, x_1, \delta) + \frac{1}{2}(c, t_1, x_2, \delta). \quad (6)$$

We now show that, with continuous time, both versions of Stochastic Impatience are equivalent to each other and correspond to ϕ being less concave than the log.

Proposition 5. *Suppose time is continuous and let (ϕ, u, D) be a KM representation of $\succsim$. The following statements are equivalent:*

1. *$\succsim$ satisfies Strong Stochastic Impatience;*

2. *$\succsim$ satisfies Stochastic Impatience';*

3. *ϕ is weakly less concave than the log.*

The results above show that not only Stochastic Impatience' and Strong Stochastic Impatience are equivalent in continuous time, but also that both are equivalent to ϕ being less concave than the log. Note that this is a stronger result than the one obtained in discrete time, where the concavity conditions were sufficient but not necessary.[16] Therefore, in continuous time, when ϕ is locally more concave than the log at some points and locally less concave than the log in some other points, a violation of Stochastic Impatience can be constructed.

The proof of Proposition 5 will be presented though two lemmas. We first show that items 1 and 3 are equivalent to each other:

Lemma 2. *Stochastic Impatience' holds if and only if ϕ is less concave than the log.*

Proof. The fact that Stochastic Impatience' holds if ϕ is less concave than the log follows by the same exact argument as in the discrete time case (see proof of part (i) of Proposition 4). We now show that Stochastic Impatience' fails if ϕ is not less concave than the log.

Let $\phi = g \circ \ln$ for some increasing function $g : u(C) \to \mathbb{R}$. Suppose g is not convex. Then, there exist $H > L$ and $\epsilon > 0$ such that

$$g\left(H + \epsilon\right) - g\left(H\right) < g\left(L + \epsilon\right) - g\left(L\right) \tag{7}$$

where $L \in u\left(C\right), \ H \in u\left(C\right), \ \left(L + \epsilon\right) \in u\left(C\right), \ \left(H + \epsilon\right) \in u\left(C\right)$.

Take the following x, y, and α:

$$y \equiv u^{-1}\left(\exp\left(H\right)\right) - \underline{x} \therefore H = \ln\left(u\left(\underline{x} + y\right)\right)$$

$$x \equiv u^{-1}\left(\exp\left(\epsilon + H\right)\right) - \underline{x} \therefore \epsilon = \ln\left(u\left(\underline{x} + x\right)\right) - \ln\left(u\left(\underline{x} + y\right)\right),$$

$$\alpha = 1 - \exp\left(L - H\right) \in (0, 1) \therefore L = \ln\left(\left(1 - \alpha\right) u\left(\underline{x} + y\right)\right).$$

Then, we have

$$g\left(H + \epsilon\right) - g\left(H\right) = g\left(\ln\left(u\left(\underline{x} + x\right)\right)\right) - g\left(\ln\left(u\left(\underline{x} + y\right)\right)\right).$$

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[16]In discrete time, Stochastic Impatience also does not imply Strong Stochastic Impatience. That is, in discrete time, ϕ weakly less concave than the log implies Stochastic Impatience, which implies Strong Stochastic Impatience, and these inclusions are strict.

Moreover,
$$g\left(L+\epsilon\right)-g\left(L\right)$$
$$=g\left(\ln\left(\left(1-\alpha\right)u\left(\underline{x}+y\right)\right)+\ln\left(u\left(\underline{x}+x\right)\right)-\ln\left(u\left(\underline{x}+y\right)\right)\right)-g\left(\ln\left(\left(1-\alpha\right)u\left(\underline{x}+y\right)\right)\right)$$
$$=g\left(\ln\left(\left(1-\alpha\right)u\left(\underline{x}+x\right)\right)\right)-g\left(\ln\left(\left(1-\alpha\right)u\left(\underline{x}+y\right)\right)\right).$$

Substituting in (7), yields

$$g\left(\ln\left(u\left(\underline{x}+x\right)\right)\right)-g\left(\ln\left(u\left(\underline{x}+y\right)\right)\right)<g\left(\ln\left(\left(1-\alpha\right)u\left(\underline{x}+x\right)\right)\right)+g\left(\ln\left(\left(1-\alpha\right)u\left(\underline{x}+y\right)\right)\right),$$

showing that Stochastic Impatience' fails if we take $c=\underline{x}$.

$\square$

Taking $\delta=+\infty$, we find that Strong Stochastic Impatience implies Stochastic Impatience', so that $1\implies 2$. By Lemma 2, $2\iff 3$. It remains to be shown that $3\implies 1$.

Lemma 3. *Suppose ϕ is less concave than the log. Then, preferences satisfy Strong Stochastic Impatience.*

Proof. Use the representation to obtain the value of $\frac{1}{2}(c,t_1,x,\delta)+\frac{1}{2}(c,t_2,y,\delta)$:

$$\frac{\phi\left(\alpha_{t_1}^\delta u_x+\left(1-\alpha_{t_1}^\delta\right)u_c\right)+\phi\left(\gamma_{t_2}^\delta u_y+\left(1-\gamma_{t_2}^\delta\right)u_c\right)}{2},$$

where $\alpha_{t_1}^\delta\equiv\frac{\int_{t\in[t_1,t_1+\delta]\cup T}D(t)dt}{\int_0^\infty D(t)dt}>\gamma_{t_2}^\delta\equiv\frac{\int_{t\in[t_2,t_2+\delta]\cup T}D(\tilde{t})d\tilde{t}}{\int_0^\infty D(\tilde{t})d\tilde{t}}$, and $u_x\equiv u(c+x)>u_y\equiv u(c+y)>u_c\equiv u(c)\geq 0$.

It suffices to show that if ϕ is less concave than the log, then

$$\phi\left(\alpha u_x+\left(1-\alpha\right)u_c\right)+\phi\left(\gamma u_y+\left(1-\gamma\right)u_c\right)$$

$$\geq\phi\left(\alpha u_y+\left(1-\alpha\right)u_c\right)+\phi\left(\gamma u_x+\left(1-\gamma\right)u_c\right)\tag{8}$$

for all $0<\gamma<\alpha<1$, $0<u_c<u_y<u_x$. Let $\phi=g\circ\ln$ for some increasing and weakly convex g, so equation (8) becomes:

$$g\left(\ln\left(\alpha u_x+\left(1-\alpha\right)u_c\right)\right)+g\left(\ln\left(\gamma u_y+\left(1-\gamma\right)u_c\right)\right)$$

$$\geq g\left(\ln\left(\alpha u_y+\left(1-\alpha\right)u_c\right)\right)+g\left(\ln\left(\gamma u_x+\left(1-\gamma\right)u_c\right)\right).\tag{9}$$

We now show that the points on the LHS of (9) have both a higher mean and a higher spread that the points on the LHS. It will then follow from the fact that g is increasing and weakly convex that the inequality holds. To see that the points on the LHS have a higher mean, note that

$$\ln\left(\alpha u_x+\left(1-\alpha\right)u_c\right)+\ln\left(\gamma u_y+\left(1-\gamma\right)u_c\right)$$

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$$\geq \ln \left(\alpha u_y + (1 - \alpha) \, u_c \right) + \ln \left(\gamma u_x + (1 - \gamma) \, u_c \right)$$

$$\iff \left[\alpha \left(1 - \gamma \right) - (1 - \alpha) \, \gamma \right] u_c u_x \geq \left[(1 - \gamma) \, \alpha - (1 - \alpha) \, \gamma \right] u_c u_y$$

$$\iff (\alpha - \gamma) \, u_c \left(u_x - u_y \right) \geq 0,$$

which is true since $\alpha > \gamma$, $u_x > u_y$, and $u_c \geq 0$. To see that the points on the LHS have a higher spread, note that

$$\ln \left(\alpha u_x + (1 - \alpha) \, u_c \right) \; > \max \left\{ \ln \left(\alpha u_y + (1 - \alpha) \, u_c \right), \; \ln \left(\gamma u_x + (1 - \gamma) \, u_c \right) \right\}$$
$$> \ln \left(\gamma u_y + (1 - \gamma) \, u_c \right).$$

Thus, condition (9) holds. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

B   Beyond Expected Utility

In this section we show that the tension between Stochastic Impatience and the separation of time and risk preferences does not rely on Expected Utility.

We extend beyond Expected Utility by assuming that preferences are at least *locally bi-linear* at $\frac{1}{2}$. This generalization includes as special cases popular models such as those of probability weighting (Rank-Dependent Utility, Quiggin 1982, and Cumulative Prospect Theory, Tversky and Kahneman 1992) and Disappointment Aversion (Gul, 1991).[17] In general, bilinearity holds if there is an increasing onto function $\pi : [0, 1] \to [0, 1]$, and a function f that evaluates (arbitrary) prizes, such that the prospect that yields x with probability α and y otherwise, with $f(x) > f(y)$, is evaluated by $\pi \left(\alpha \right) f \left(x \right) + \left[1 - \pi \left(\alpha \right) \right] f \left(y \right)$. Since our goal is to be as general as possible, we only require preferences to be bilinear for equally likely binary lotteries $(\alpha = \frac{1}{2})$ – the *local* bilinear model (Dean and Ortoleva, 2017).[18]Applying it to our setting, we obtain the following generalization of the KM model using the continuous time setup of Appendix A.

Definition 6. We say that $\succcurlyeq$ admits a *local bilinear KM* representation if there exist strictly increasing and continuous $u : [\underline{x}, \overline{x}] \to \mathbb{R}_+$ with $u(\underline{x}) = 0$, a strictly decreasing $D : T \to [0, 1]$, a strictly increasing and differentiable $\phi : u([\underline{x}, \overline{x}]) \to \mathbb{R}$, and $\pi(\frac{1}{2}) \in (0, 1)$, such that for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, $p = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$ with $\int_0^{\bar{t}} D(t)u(x(t))dt \geq \int_0^{\bar{t}} D(t)u(y(t))dt$ is evaluated according to:

$$V(p) = \pi \left(\frac{1}{2} \right) \phi \left(\frac{\int_0^{\bar{t}} D(t)u(x(t))dt}{\int_0^{\bar{t}} D(t)dt} \right) + \left[1 - \pi \left(\frac{1}{2} \right) \right] \phi \left(\frac{\int_0^{\bar{t}} D(t)u(y(t))dt}{\int_0^{\bar{t}} D(t)dt} \right).$$

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[17]It also allows for generalizations of Rank-Dependent Expected Utility, e.g., the minimum from a set of probability distortions (Dean and Ortoleva, 2017). On the other hand, it does not encompass all known models of risk preferences (e.g., it does not encompass Cautious Expected Utility, Cerreia-Vioglio et al. 2015).

[18]This is a local specification of the bilinear (or biseparable) model of Ghirardato and Marinacci (2001) for objective risk. Here, preferences are not restricted to be Bilinear in general, but only that there is some bilinear representation for 50/50 lotteries.

It is easy to see that in a local bilinear KM representation Residual Risk Aversion can be achieved either by adding curvature to ϕ, as in the standard KM representation, or by adding non-Expected Utility and First Order Risk Aversion (Segal and Spivak, 1990) by positing that $\pi(\frac{1}{2}) < \frac{1}{2}$ – underweighting the best outcome.

Proposition 6. *Let $\succcurlyeq$ be a preference relation over Δ that admits a local bilinear KM representation (u, D, ϕ, π). If $\pi(\frac{1}{2}) < \frac{1}{2}$, then $\succcurlyeq$ violates Stochastic Impatience'.*

The result above shows that in a continuous time setting, even if we go beyond Expected Utility by looking at the broad class of local bilinear models, displaying First Order Risk Aversion *always* leads to violations of Stochastic Impatience, independently of the shape of ϕ. Intuitively, this derives from the fact that First Order Risk Aversion implies extreme amounts of risk aversion in a neighborhood around certainty, and we have already seen how Stochastic Impatience is violated once risk aversion towards discounted utilities is high enough.

Proof. Take $c = \underline{x}$ (so that $u(\underline{x}) = 0$), $t_1 = 0$, and let $x_1 \in \text{int}(C)$ be such that $\phi'(u(x_1)) > 0$ (which exists because ϕ is differentiable and strictly increasing). Fix $t_2 > 0$ and let x_2 be such that:

$$u(x_2) = u(x_1) \frac{\int_{t_2}^{\bar{t}} D(t)dt}{\int_0^{\bar{t}} D(t)dt}. \tag{10}$$

Note that $x_2 \in \text{int}(C)$ because $0 < u(x_2) < u(x_1)$. Thus, by construction, $(c, t_2, x_1) \sim (c, t_1, x_2)$.

We will show that, if t_2 is close enough to 0, then

$$\pi\left(\frac{1}{2}\right)\phi(u(x_1)) + (1 - \pi\left(\frac{1}{2}\right))\phi\left(u(x_2)\frac{\int_{t_2}^{\bar{t}} D(t)dt}{\int_0^{\bar{t}} D(t)dt}\right)$$

$$= \pi\left(\frac{1}{2}\right)\phi(u(x_1)) + (1 - \pi\left(\frac{1}{2}\right))\phi\left(u(x_1)\left(\frac{\int_{t_2}^{\bar{t}} D(t)dt}{\int_0^{\bar{t}} D(t)dt}\right)^2\right)$$

$$< \phi\left(u(x_1)\frac{\int_{t_2}^{\bar{t}} D(t)dt}{\int_0^{\bar{t}} D(t)dt}\right)$$

where the equality above uses (10). First note that both sides equal $\phi(u(x_1))$ for $t_2 = 0$. We now show that the LHS falls faster than RHS when we increase t_2 slightly, generating a violation of Stochastic Impatience. By Leibniz's rule we have

$$\frac{\partial}{\partial t_2}\phi\left(u(x_1)\frac{\int_{t_2}^{\bar{t}} D(t)dt}{\int_0^{\bar{t}} D(t)dt}\right)\Bigg|_{t_2=0} = -\phi'(u(x_1))\frac{D(0)}{\int_0^{\bar{t}} D(t)dt}$$

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whereas

$$
\frac{\partial}{\partial t_2}\left[\pi\left(\frac{1}{2}\right)\phi\left(u(x_1)\right)+\left(1-\pi\left(\frac{1}{2}\right)\right)\phi\left(u\left(x_1\right)\left(\frac{\int_{t_2}^{\bar{t}}D(t)dt}{\int_0^{\bar{t}}D(t)dt}\right)^2\right)\right]_{|t_2=0}
$$

$$
= -\left(1-\pi\left(\frac{1}{2}\right)\right)\phi'\left(u\left(x_1\right)\right)\frac{2D(0)}{\int_0^{\bar{t}}D(t)dt}
$$

So we want to show that

$$
\frac{-\phi'\left(u(x_1)\right)D(0)}{\int_0^{\bar{t}}D(t)dt}>-\left[1-\pi\left(\frac{1}{2}\right)\right]\phi'\left(u\left(x_1\right)\right)\frac{2D(0)}{\int_0^{\bar{t}}D(t)dt}
$$

Since $\frac{\phi'(u(x_1))D(0)}{\int_0^{\bar{t}}D(t)dt}>0$, this is true if and only if $\pi(\frac{1}{2})<\frac{1}{2}$.

$\square$

C   Proofs of results in the text

C.1   Proof of Observation 1

Using the representation, Stochastic Impatience holds if and only if

$$
\frac{D(t_1)u(x_1)+D(t_2)u(x_2)}{2}+\sum_{t\notin\{t_1,t_2\}}D(t)u(c)\geq\frac{D(t_1)u(x_2)+D(t_2)u(x_1)}{2}+\sum_{t\notin\{t_1,t_2\}}D(t)u(c)
$$

for all all $t_1<t_2$, c, and $x_1>x_2$. Rearrange this expression to obtain:

$$
[D(t_1)-D(t_2)]\,[u(x_1)-u(x_2)]\geq 0.
$$

Since u is strictly increasing, this inequality holds if and only if $D(t_1)\geq D(t_2)$.   ∎

C.2   Proof of Proposition 1

Let $\rho\equiv\frac{1}{\psi}$ denote the inverse of EIS. For notational simplicity, we will work with ρ instead of ψ.

   Since preferences are dynamically consistent, it suffices to look at lotteries in which the earliest payment is made in period 1. Consider a lottery that, with equal probability, either starts paying an increment of x in period 1 or starts paying an increment of y in period t: $\frac{1}{2}(c,1,c+x)+\frac{1}{2}(c,t,c+y)$.

   Note that $\succcurlyeq$ satisfies Stochastic Impatience (SI) if:

$$
\frac{1}{2}(c,1,c+x)+\frac{1}{2}(c,t,c+y)\succcurlyeq\frac{1}{2}(c,1,c+y)+\frac{1}{2}(c,t,c+x)
$$

for any $c>0$, any $x>y$, and any $t\in\{2,3,...\}$. The proof will be given through a series of lemmas.

Lemma 4. *The value of lottery $\frac{1}{2}(c, 1, c + x) + \frac{1}{2}(c, t, c + y)$ is*

$$\left\{ (1 - \beta) c^{1-\rho} + \beta \left[\frac{(c + x)^{1-\alpha} + \left\{ c^{1-\rho} + \beta^{t-1} \left[(c + y)^{1-\rho} - c^{1-\rho} \right] \right\}^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}.$$

Proof. Recall that in EZ, lotteries are evaluated using the recursion:

$$V_t = \left\{ (1 - \beta) x(t)^{1-\rho} + \beta \left[E_t \left(V_{t+1}^{1-\alpha} \right) \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}. \tag{11}$$

With 50% chance, the individual gets $c + x$ in all future periods, giving a continuation value of $V_1 = c + x$. With 50% chance, consumption equals c up to period $t - 1$, after which the individual gets $c + y$. Therefore, $V_t = c + y$. Proceeding backwards, we obtain:

$$V_{t-n} = \left\{ (1 - \beta) c^{1-\rho} \left(1 + \beta + ... + \beta^{n-1} \right) + \beta^n (c + y)^{1-\rho} \right\}^{\frac{1}{1-\rho}},$$

for any $n = 1, ..., t - 1$. In particular,

$$
\begin{aligned}
V_1 &= \left\{ (1 - \beta) c^{1-\rho} \left(1 + \beta + ... + \beta^{t-2} \right) + \beta^{t-1} (c + y)^{1-\rho} \right\}^{\frac{1}{1-\rho}} \\
&= \left\{ c^{1-\rho} + \beta^{t-1} \left[(c + y)^{1-\rho} - c^{1-\rho} \right] \right\}^{\frac{1}{1-\rho}}
\end{aligned}
,$$

where the second line uses the fact that $1 + \beta + ... + \beta^{t-2} = \frac{1 - \beta^{t-1}}{1-\beta}$.

Moving to period 0, we obtain:

$$V_0 = \left\{ (1 - \beta) c^{1-\rho} + \beta \left[\frac{(c + x)^{1-\alpha} + \left\{ c^{1-\rho} + \beta^{t-1} \left[(c + y)^{1-\rho} - c^{1-\rho} \right] \right\}^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}.$$

$\square$

By the homotheticity of EZ preferences, we can, without loss of generality, take a background consumption of $c = 1$. Therefore, Stochastic Impatience holds if and only if, for all $z_H > z_L > 1$ and all $t \in \{2, 3, ...\}$, we have

$$
\begin{aligned}
&\left\{ 1 - \beta + \beta \left[\frac{z_H^{1-\alpha} + \left[1 + \beta^{t-1} \left(z_L^{1-\rho} - 1 \right) \right]^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}} \\
&\geq \left\{ 1 - \beta + \beta \left[\frac{z_L^{1-\alpha} + \left[1 + \beta^{t-1} \left(z_H^{1-\rho} - 1 \right) \right]^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}
\end{aligned}
. \tag{12}
$$

Lemma 5. *Stochastic Impatience holds if and only if*

$$z^{\rho-\alpha} \geq \left[1 + \beta^{t-1}\left(z^{1-\rho} - 1\right)\right]^{\frac{\rho-\alpha}{1-\rho}} \beta^{t-1} \tag{13}$$

for all $z \geq 1$ and all $t = \{2, 3, ...\}$.

Proof. Let $\Phi : [1, +\infty) \to \mathbb{R}$ be given by

$$\Phi(z) \equiv z^{1-\alpha} - \left[1 + \beta^{t-1}\left(z^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}} .$$

The proof has two parts. In the first part, we show that Stochastic Impatience holds if and only if:

- $\Phi'(z) \geq 0$ for all $z > 1$ if $\alpha < 1$.

- $\Phi'(z) \leq 0$ for all $z > 1$ if $\alpha > 1$.

To establish this result, we rearrange inequality (12) in each of 4 possible cases.
  Case 1: $\alpha, \rho < 1$.

$$z_H^{1-\alpha} + \left[1 + \beta^{t-1}\left(z_L^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}} \geq z_L^{1-\alpha} + \left[1 + \beta^{t-1}\left(z_H^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}}$$

$$\iff z_H^{1-\alpha} - \left[1 + \beta^{t-1}\left(z_H^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}} \geq z_L^{1-\alpha} - \left[1 + \beta^{t-1}\left(z_L^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}} .$$

Case 2: $\alpha, \rho > 1$.

$$z_H^{1-\alpha} + \left[1 + \beta^{t-1}\left(z_L^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}} \leq z_L^{1-\alpha} + \left[1 + \beta^{t-1}\left(z_H^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}}$$

$$\iff z_H^{1-\alpha} - \left[1 + \beta^{t-1}\left(z_H^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}} \leq z_L^{1-\alpha} - \left[1 + \beta^{t-1}\left(z_L^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}} .$$

Case 3: $\alpha > 1 > \rho$.

$$z_H^{1-\alpha} - \left[1 + \beta^{t-1}\left(z_H^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}} \leq z_L^{1-\alpha} - \left[1 + \beta^{t-1}\left(z_L^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}} .$$

Case 4: $\rho > 1 > \alpha$.

$$z_H^{1-\alpha} - \left[1 + \beta^{t-1}\left(z_H^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}} \geq z_L^{1-\alpha} - \left[1 + \beta^{t-1}\left(z_L^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}} .$$

In the second part, we differentiate Φ to obtain:

$$\Phi'(z) = (1-\alpha)\left\{z^{-\alpha} - \left[1 + \beta^{t-1}\left(z^{1-\rho} - 1\right)\right]^{\frac{\rho-\alpha}{1-\rho}} \beta^{t-1} z^{-\rho}\right\} .$$

Since, by the previous result, Stochastic Impatience holds if $\Phi'(z) \geq 0$ when $\alpha < 1$ and $\Phi'(z) \leq 0$ if $\alpha > 1$, it follows that the term inside brackets must be weakly positive for all $z \geq 1$:

$$z^{-\alpha} \geq \left[1 + \beta^{t-1}\left(z^{1-\rho} - 1\right)\right]^{\frac{\rho-\alpha}{1-\rho}} \beta^{t-1} z^{-\rho}.$$

Multiplying both sides by $z^\rho > 0$, we obtain (13). $\qquad\square$

We now use Lemma 5 to determine when Stochastic Impatience holds.

Lemma 6. *Stochastic Impatience if and only if either (i) $\alpha \leq \frac{1}{\psi}$, or (ii) $\alpha < 1$.*

Proof. The proof considers each of the six possible cases. We start with the two cases under which Stochastic Impatience fails:

Case 1. $\alpha > 1 > \rho$**.** To show that Stochastic Impatience fails in this case, note that after some algebraic manipulations, we can rewrite (13) as:

$$\frac{1 - \beta^{t-1}}{z^{1-\rho}} \geq \beta^{(t-1)\frac{1-\rho}{\alpha-\rho}} - \beta^{t-1}$$

for all $z \geq 1$. Note that the LHS converges to zero as $z \nearrow +\infty$. Moreover, the LHS is bounded away from zero since

$$\beta^{(t-1)\frac{1-\rho}{\alpha-\rho}} > \beta^{t-1} \iff (t-1)\frac{1-\rho}{\alpha-\rho} < t - 1 \iff 1 < \alpha.$$

Therefore, there exists $\bar{z}$ such that (13) fails for all $z > \bar{z}$, showing that Stochastic Impatience fails.

Case 2. $\alpha > \rho > 1$. To show that Stochastic Impatience fails, rearrange (13) as

$$\beta^{\frac{(t-1)(\rho-1)}{\alpha-\rho}} \geq \left(1 - \beta^{t-1}\right) z^{\rho-1} + \beta^{t-1}.$$

Note that, as $z \searrow 1$, the RHS converges to 1, whereas the LHS is always strictly less than 1 (since $\frac{(t-1)(\rho-1)}{\alpha-\rho} > 0$). Therefore, this inequality fails for z close to 1, showing that Stochastic Impatience fails.

We now turn to the cases where Stochastic Impatience holds:

Case 3. $1 > \alpha > \rho$. Rearranging (13), we find that Stochastic Impatience holds if and only if

$$\left[1 + \beta^{t-1}\left(z^{1-\rho} - 1\right)\right]^{\frac{\alpha-\rho}{1-\rho}} \geq z^{\alpha-\rho}\beta^{t-1}$$

for all $z > 1$. Rearrange this inequality as

$$\frac{1 - \beta^{t-1}}{z^{1-\rho}} + \beta^{t-1} \geq \beta^{\frac{(t-1)(1-\rho)}{\alpha-\rho}}.$$

Because LHS is decreasing in z and converges to β^{t-1} as $z \nearrow +\infty$, this condition holds for all $z > 1$ if and only if

$$\beta^{t-1} \geq \beta^{(t-1)\left(\frac{1-\rho}{\alpha-\rho}\right)},$$

which is true since $\rho < \alpha \leq 1$. Thus, Stochastic Impatience holds in this case.

Case 4. $1 < \alpha < \rho$. Rewrite (13) as

$$z^{\rho-1} \cdot \left(1 - \beta^{t-1}\right) + \beta^{t-1} \geq \beta^{\frac{(t-1)(\rho-1)}{\rho-\alpha}}.$$

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Since the LHS is increasing in z, this inequality holds for all $z > 1$ if and only if it holds for $z = 1$:

$$1 \geq \beta^{\frac{(t-1)(\rho-1)}{\rho-\alpha}},$$

which is true because $\frac{(t-1)(\rho-1)}{\rho-\alpha} > 0$. Thus, Stochastic Impatience also holds in this case.

Case 5. $\alpha < \rho < 1$. Rewrite (13) as

$$\frac{1}{\beta^{\frac{(t-1)(1-\rho)}{\rho-\alpha}}} \geq \frac{1-\beta^{t-1}}{z^{1-\rho}} + \beta^{t-1}.$$

Since the RHS is decreasing in z, this condition holds for all $z > 1$ if and only if it holds for $z = 1$:

$$\frac{1}{\beta^{\frac{(t-1)(1-\rho)}{\rho-\alpha}}} \geq 1 - \beta^{t-1} + \beta^{t-1} \iff 1 \geq \beta^{\frac{(t-1)(1-\rho)}{\rho-\alpha}},$$

which is true since $\frac{(t-1)(1-\rho)}{\rho-\alpha} \geq 0$ under the parameters above.

Case 6. $\alpha < 1 < \rho$. Notice that (13) can be simplified as

$$z^{\rho-1}\left(1-\beta^{t-1}\right) + \beta^{t-1} \geq \beta^{(t-1)\left(\frac{\rho-1}{\rho-\alpha}\right)}.$$

Since the LHS is increasing in z, this condition holds for all $z > 1$ if and only if it holds for $z = 1$:

$$1 \geq \beta^{(t-1)\left(\frac{\rho-1}{\rho-\alpha}\right)},$$

which is true since $(t-1)\left(\frac{\rho-1}{\rho-\alpha}\right) > 0$. $\qquad\square$

C.3   Proof of Proposition 2

The utility of the lottery $\frac{1}{2}(c, 1, c+x) + \frac{1}{2}(c, t, c+y)$ is

$$-\frac{\exp\left(-k\frac{\beta}{1-\beta}u\left(c+x\right)\right) + \exp\left(-k \cdot \left(\frac{\beta-\beta^t}{1-\beta}u\left(c\right) + \frac{\beta^t}{1-\beta}u\left(c+y\right)\right)\right)}{2}.$$

Therefore, Stochastic Impatience fails if and only if there exist $x > y > c$ and $t \in \{2, 3, ...\}$ such that

$$-\frac{\exp\left(-k\frac{\beta}{1-\beta}u\left(c+y\right)\right) + \exp\left(-k \cdot \left(u\left(c\right)\frac{\beta-\beta^t}{1-\beta} + u\left(c+x\right)\frac{\beta^t}{1-\beta}\right)\right)}{2}$$

$$> -\frac{\exp\left(-k\frac{\beta}{1-\beta}u\left(c+x\right)\right) + \exp\left(-k \cdot \left(u\left(c\right)\frac{\beta-\beta^t}{1-\beta} + u\left(c+y\right)\frac{\beta^t}{1-\beta}\right)\right)}{2}.$$

Rearrange this inequality as:

$$\exp\left(-k\frac{\beta}{1-\beta}u\left(c+y\right)\right) - \exp\left(-k\cdot\left(u\left(c\right)\frac{\beta-\beta^t}{1-\beta}+u\left(c+y\right)\frac{\beta^t}{1-\beta}\right)\right)$$

$$< \exp\left(-k\frac{\beta}{1-\beta}u\left(c+x\right)\right) - \exp\left(-k\cdot\left(u\left(c\right)\frac{\beta-\beta^t}{1-\beta}+u\left(c+x\right)\frac{\beta^t}{1-\beta}\right)\right).$$

For simplicity, we will assume that u is a differentiable function, although it is immediate to generalize the argument for when it is not. Then, Stochastic Impatience fails if and only if there exist $x > c$ and $t \in \{2, 3, ...\}$ such that

$$\frac{d}{dx}\left[\exp\left(-k\frac{\beta}{1-\beta}u\left(c+x\right)\right) - \exp\left(-k\cdot\left(u\left(c\right)\frac{\beta-\beta^t}{1-\beta}+u\left(c+x\right)\frac{\beta^t}{1-\beta}\right)\right)\right] > 0$$

Evaluating the derivative, the previous inequality becomes

$$\exp\left(-k\frac{\beta}{1-\beta}u\left(c+x\right)\right) < \beta^{t-1}\exp\left(-k\cdot\left(u\left(c\right)\frac{\beta-\beta^t}{1-\beta}+u\left(c+x\right)\frac{\beta^t}{1-\beta}\right)\right).$$

Since both sides are positive, we can take logs to obtain:

$$-k\frac{\beta}{1-\beta}u\left(c+x\right) < (t-1)\ln\beta - k\cdot\left(u\left(c\right)\frac{\beta-\beta^t}{1-\beta}+u\left(c+x\right)\frac{\beta^t}{1-\beta}\right),$$

which can be rearranged as:

$$u\left(c+x\right) - u\left(c\right) > \frac{(t-1)(1-\beta)}{\beta-\beta^t}\cdot\frac{-\ln\beta}{k}. \tag{14}$$

Therefore, Stochastic Impatience fails if and only if there exist $x > c$ and $t = \{2, 3, ...\}$ such that (14) holds. For $t = 2$, condition (14) becomes

$$u\left(c+x\right) - u\left(c\right) > \frac{-\ln\beta}{\beta\cdot k}.$$

To complete the proof, we verify that this inequality holds for some t if and only if it holds for $t = 2$, that is:

$$\frac{-\ln\beta}{\beta\cdot k} \leq \frac{(t-1)(1-\beta)}{\beta-\beta^t}\cdot\frac{-\ln\beta}{k}$$

for all $t > 2$. To see this, rearrange the inequality above as

$$(t-1)(1-\beta) - 1 + \beta^{t-1} \geq 0. \tag{15}$$

At $\beta = 1$, both sides equal zero. The derivative of the expression on the RHS with respect to β is:

$$-(t-1) + (t-1)\beta^{t-2} = -(t-1)\left(1-\beta^{t-2}\right),$$

which is strictly negative for all $\beta \in [0, 1)$ and all $t > 2$. Thus, (15) holds for all $\beta \in [0, 1]$, concluding the proof. ■

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C.4   Proof of Lemma 1

For necessity, note that, when restricted to degenerate streams, the representation is a monotone transformation of $\sum_T D(t)u(x(t))$, so preferences must satisfy Assumption 1. Moreover, since risky lotteries as evaluated by taking expectations, preferences satisfy Assumption 2 as in Expected Utility Theory.

For sufficiency, first note that by Assumption 1, there exist a strictly increasing and continuous $\tilde{u} : [\underline{x}, \overline{x}] \to \mathbb{R}_+$ and a strictly decreasing $D : T \to [0,1]$ such that $\succcurlyeq$ restricted to $\mathcal{X}$ are represented by

$$F^*(\mathbf{x}) := \sum_{t \in T} D(t)\tilde{u}(x(t)).$$

Applying a positive transformation, it follows that the same preference must also be represented by

$$F(\mathbf{x}) := \frac{1}{\sum_T D(t)} \sum_T D(t)[\tilde{u}(x(t)) - \tilde{u}(\underline{x})] = \frac{1}{\sum_T D(t)} \sum_T D(t)u(x(t)),$$

where $u(x(t)) \equiv \tilde{u}(x(t)) - \tilde{u}(\underline{x})$, so that $u(\underline{x}) = \tilde{u}(\underline{x}) - \tilde{u}(\underline{x}) = 0$. Note that $F(\mathcal{X}) = [0, u(\overline{x})]$.

By Assumption 2, there exists $U : \mathcal{X} \to \mathbb{R}$ such that $\succcurlyeq$ is represented by

$$V(p) := \mathbb{E}_p\big[U\big].$$

It follows that U and F represent the same preferences over $\mathcal{X}$, i.e., for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$,

$$U(\mathbf{x}) \geq U(\mathbf{y}) \Leftrightarrow \mathbf{x} \succcurlyeq \mathbf{y} \Leftrightarrow F(\mathbf{x}) \geq F(\mathbf{y}). \tag{16}$$

Therefore, there must exist an increasing $\phi : [0, u(\overline{x})] \to \mathbb{R}$ such that $U = \phi \circ F$.

We claim that ϕ must be strictly increasing. Suppose not. Then, there are $a, b \in [0, u(\overline{x})]$ with $a > b$ and $\phi(a) = \phi(b)$. Consider the streams $\mathbf{x}$ and $\mathbf{y}$ that return $u^{-1}(a)$ and $u^{-1}(b)$ each period, respectively. Since $a > b$ we must have $F(\mathbf{x}) = a > b = F(\mathbf{y})$. At the same time, since $\phi(a) = \phi(b)$, we have $U(\mathbf{x}) = \phi(F(\mathbf{x})) = \phi(a) = \phi(b) = \phi(F(\mathbf{y})) = U(\mathbf{y})$, violating (16).

The uniqueness claims follow from the same arguments as in the Expected Utility Theorem. ∎

C.5   Proof of Proposition 3

The proof will be presented in three lemmas.

Lemma 7. *Preferences are Residual Risk Averse (Seeking) if for all $x, y \in [0, u(\overline{x})]$,*

$$\phi\left(\gamma x + (1 - \gamma)\, y\right) + \phi\left(\gamma y + (1 - \gamma)\, x\right) \geq (\leq)\phi\left(x\right) + \phi\left(y\right) \tag{17}$$

where $\gamma \equiv \frac{D(1)}{D(1)+D(2)} \in \left(\frac{1}{2}, 1\right)$.

Proof. Note that by Definition 2 and the KM representation, preferences display Residual Risk Aversion whenever:

$$\phi\left(\frac{D(1)u(a) + D(2)u(d) + \sum_{t\notin\{1,2\}} D(t)u(x)}{\sum_T D(t)}\right) = \phi\left(\frac{[D(1) + D(2)]\,u(b) + \sum_{t\notin\{1,2\}} D(t)u(x)}{\sum_T D(t)}\right)$$

and

$$\phi\left(\frac{D(1)u(d) + D(2)u(a) + \sum_{t\notin\{1,2\}} D(t)u(x)}{\sum_T D(t)}\right) = \phi\left(\frac{[D(1) + D(2)]\,u(c) + \sum_{t\notin\{1,2\}} D(t)u(x)}{\sum_T D(t)}\right)$$

imply

$$\frac{\phi\left(u(b)\right) + \phi\left(u(c)\right)}{2} \geq \frac{\phi\left(u(a)\right) + \phi\left(u(d)\right)}{2}.$$

Since ϕ is strictly increasing, the first two equations can be simplified as:

$$u(b) = \frac{D(1)u(a) + D(2)u(d)}{D(1) + D(2)} \text{ and } u(c) = \frac{D(1)u(d) + D(2)u(a)}{D(1) + D(2)}.$$

Therefore, Residual Risk Aversion holds if and only if, for all a and all d,

$$\phi\left(\frac{D(1)u(a) + D(2)u(d)}{D(1) + D(2)}\right) + \phi\left(\frac{D(1)u(d) + D(2)u(a)}{D(1) + D(2)}\right) \geq \phi\left(u(a)\right) + \phi\left(u(d)\right). \quad (18)$$

Letting $\gamma \equiv \frac{D(1)}{D(1)+D(2)}$, $x \equiv u(a)$, and $y \equiv u(d)$ concludes the proof. $\qquad\square$

Lemma 8. *Let (ϕ, u, D) be a KM representation of $\succcurlyeq$.*

- *If ϕ is discontinuous at any point $x \neq 0$, then $\succcurlyeq$ is not Residual Risk Averse.*

- *If ϕ is discontinuous at any point $x \neq u\left(\overline{x}\right)$, then $\succcurlyeq$ is not Residual Risk Seeking.*

Proof. Suppose ϕ is discontinuous at $x > 0$. Let $\{h_t\} \searrow x$ be a decreasing sequence that converges to x, let $\{l_t\} \nearrow x$ be a non-decreasing sequence that converges to x. Since $\{\phi(h_t)\}$ and $\{\phi(l_t)\}$ are monotone and bounded, they converge. Let

$$\phi_+ := \lim_{t\to\infty} \phi(h_t) > \lim_{t\to\infty} \phi(l_t) = \phi_-.$$

For each t, let $u_{a_t} := h_t$ and $u_{d_t} := \frac{l_t - \gamma h_t}{1-\gamma}$. Note that

$$u_{d_t} < \gamma u_{d_t} + (1-\gamma)\,u_{a_t} < \gamma u_{a_t} + (1-\gamma)\,u_{d_t} = l_t < x < h_t = u_{a_t}.$$

Since ϕ is bounded (by $\phi(0)$ and $\phi(u(\overline{x}))$), we can assume that the sequences $\{\phi\left(\gamma u_{a_t} + (1-\gamma)\,u_{d_t}\right)\}$, $\{\phi\left(\gamma u_{d_t} + (1-\gamma)\,u_{a_t}\right)\}$, $\{\phi\left(u_{a_t}\right)\}$, and $\{\phi\left(u_{d_t}\right)\}$ are convergent (taking a subsequence if necessary). Therefore,

$$\lim_{t\to\infty} \phi(u_{d_t}) = \lim_{t\to\infty} \phi\left(\gamma u_{d_t} + (1-\gamma)\,u_{a_t}\right) = \lim_{t\to\infty} \phi\left(\gamma u_{a_t} + (1-\gamma)\,u_{d_t}\right) = \phi_-,$$

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and
$$\lim_{t\to\infty} \phi(u_{a_t}) = \phi_+ > \phi_-.$$

Therefore, there exists $\bar{t}$ such that for all $t > \bar{t}$,

$$\phi\left(\gamma u_{a_t} + (1-\gamma)\, u_{d_t}\right) + \phi\left(\gamma u_{d_t} + (1-\gamma)\, u_{a_t}\right) < \phi\left(u_{a_t}\right) + \phi\left(u_{d_t}\right),$$

which, by (17), shows that preferences are not Residual Risk Averse.

Next, suppose ϕ is discontinuous at $x < u\,(\bar{x})$. Let $\{h_t\} \searrow x$ be a non-increasing sequence that converges to x, let $\{l_t\} \nearrow x$ be an increasing sequence that converges to x. As before, let

$$\phi_+ := \lim_{t\to\infty} \phi(h_t) > \lim_{t\to\infty} \phi(l_t) = \phi_-,$$

where the limits exist by the Monotone Convergence Theorem.

For each t, take $u_{d_t} := l_t$ and take $u_{a_t} = \frac{h_t - \gamma l_t}{1-\gamma}$. Note that

$$u_{a_t} > \gamma u_{a_t} + (1-\gamma)\, u_{d_t} > \gamma u_{d_t} + (1-\gamma)\, u_{a_t} = h_t > x > l_t = u_{d_t}.$$

As before (taking a subsequence if necessary), we have

$$\lim_{t\to\infty} \phi\left(u_{a_t}\right) = \lim_{t\to\infty} \phi\left(\gamma u_{a_t} + (1-\gamma)\, u_{d_t}\right) = \lim_{t\to\infty} \phi\left(\gamma u_{d_t} + (1-\gamma)\, u_{a_t}\right) = \phi_+,$$

and

$$\lim_{t\to\infty} \phi\left(u_{d_t}\right) = \phi_- < \phi_+.$$

Thus, there exists $\bar{t}$ such that for all $t > \bar{t}$,

$$\phi\left(\gamma u_{a_t} + (1-\gamma)\, u_{d_t}\right) + \phi\left(\gamma u_{d_t} + (1-\gamma)\, u_{a_t}\right) > \phi\left(u_{a_t}\right) + \phi\left(u_{d_t}\right),$$

showing that preferences are not Residual Risk Seeking. □

Lemma 9. *Let (ϕ, u, D) be a KM representation of $\succsim$.*

- *$\succsim$ is Residual Risk Averse if and only if ϕ is concave.*

- *$\succsim$ is Residual Risk Seeking if and only if ϕ is convex.*

Proof. To establish sufficiency, suppose, without loss of generality, that $x > y$, so that

$$x > \gamma x + (1-\gamma)y > \gamma y + (1-\gamma)x > y.$$

It follows from the definition of concavity (convexity) and inequality (17) that preferences are Residual Risk Averse (Seeking) if ϕ is concave (convex). We now establish necessity.

Suppose preferences are Residual Risk Averse. By the previous lemma, ϕ must be continuous at any point $x > 0$. We need to show that ϕ is concave. Suppose not. Then, there exist $x, y \in [0, u(\overline{x})]$ with $x > y$ and $t \in (0, 1)$ such that

$$t\phi(x) + (1 - t)\,\phi(y) > \phi\left(tx + (1 - t)\,y\right). \tag{19}$$

Let $F : [0, 1] \to \mathbb{R}$ given by

$$F(\tilde{t}) \equiv \phi\left(\tilde{t}x + \left(1 - \tilde{t}\right)y\right) - \left[\tilde{t}\phi(x) + \left(1 - \tilde{t}\right)\phi(y)\right],$$

and note that $F(t) < 0$. Since ϕ can only be discontinuous at 0, $F(\tilde{t})$ is continuous at all $\tilde{t} > 0$. It is continuous at $\tilde{t} = 0$ if either $y > 0$ or if ϕ is continuous at 0.

Let

$$L \equiv \left\{\tilde{t} \in [0, t] : \ F(\tilde{t}) \leq 0\right\} \ \text{and} \ H \equiv \left\{\tilde{t} \in [t, 1] : \ F(\tilde{t}) \leq 0\right\}.$$

Let

$$l \equiv \sup L \ \text{and} \ h \equiv \inf H.$$

Because $F(\tilde{t})$ is continuous at $\tilde{t} > 0$ and $F(t) < 0$, it follows that $l < t < h$. Moreover, it follows from the definitions of the supremum and infimum that

$$F\left(\tilde{t}\right) < 0 \ \forall \tilde{t} \in (l, h).$$

We claim that $F(l) = 0$. There are two cases to consider. If F is continuous at 0, then L is a compact and non-empty set ($0 \in L$), which implies that $F(l) = 0$. Suppose, instead, that F is discontinuous at 0, which can only happen if $y = 0$ and ϕ is discontinuous at 0. Because ϕ is increasing, the discontinuity must correspond to an upwards jump: $\phi(0) < \lim_{z \searrow 0} \phi(z) =: \phi(0_+)$. Since

$$\lim_{\tilde{t} \searrow 0} F(\tilde{t}) = \lim_{\tilde{t} \searrow 0} \left\{\phi\left(\tilde{t}x\right) - \left[\tilde{t}\phi(x) + \left(1 - \tilde{t}\right)\phi(0)\right]\right\} = \phi\left(0_+\right) - \phi(0) > 0,$$

and F is continuous at $\tilde{t} > 0$, there exists $\overline{t} > 0$ such that $F(\tilde{t}) > 0$ for all $\tilde{t} \in (0, \overline{t})$. Hence, $l \geq \overline{t}$ by the definition of supremum. Therefore,

$$l \equiv \sup L = \sup \left\{\tilde{t} \in [\overline{t}, t] : \ F(\tilde{t}) \geq 0\right\}.$$

Because $\left\{\tilde{t} \in [\overline{t}, t] : \ F(\tilde{t}) \geq 0\right\}$ is compact ($F(\tilde{t})$ is continuous for all $\tilde{t} > 0$) and non-empty ($\overline{t}$ belongs to it), it again follows that $F(l) = 0$.

Next, we show that $F(h) = 0$. Because $F(t) < 0$ and $F(0) = 0$, (19) implies that $t > 0$. Therefore, $F(\tilde{t})$ is continuous at $[t, 1]$, implying that H is a compact set. Because it is also non-empty ($1 \in H$), we must have $F(h) = 0$.

Substituting the definition of F, we have shown:

$$l\phi(x) + (1 - l)\,\phi(y) = \phi\left(lx + (1 - l)\,y\right), \tag{20}$$

$$h\phi(x) + (1 - h)\,\phi(y) = \phi\left(hx + (1 - h)\,y\right), \tag{21}$$

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and

$$\tilde{t}\phi(x) + \left(1 - \tilde{t}\right)\phi(y) > \phi\left(\tilde{t}x + \left(1 - \tilde{t}\right)y\right) \tag{22}$$

for all $\tilde{t} \in (l, h)$.

Let $y' \equiv lx + (1-l)y$ and $x' \equiv hx + (1-h)y$, so that $y < y' < x' < x$. Note that, for all $\lambda \in (0, 1)$, we have

$$\begin{aligned}\lambda y' + (1-\lambda)x' &= \lambda\left[lx + (1-l)y\right] + (1-\lambda)\left[hx + (1-h)y\right] \\ &= \left[\lambda l + (1-\lambda)h\right]x + \{1 - \left[\lambda l + (1-\lambda)h\right]\}y\end{aligned} \tag{23}$$

Since $\lambda l + (1-\lambda)h \in (l, h)$, we have

$$\begin{aligned}\phi\left(\lambda y' + (1-\lambda)x'\right) &= \phi\left(\left[\lambda l + (1-\lambda)h\right]x + \{1 - \left[\lambda l + (1-\lambda)h\right]\}y\right) \\ &< \left[\lambda l + (1-\lambda)h\right]\phi(x) + \{1 - \left[\lambda l + (1-\lambda)h\right]\}\phi(y) \\ &= \lambda\left[l\phi(x) + (1-l)\phi(y)\right] + (1-\lambda)\left[h\phi(x) + (1-h)\phi(y)\right] \\ &= \lambda\phi\left(lx + (1-l)y\right) + (1-\lambda)\phi\left(hx + (1-h)y\right) \\ &= \lambda\phi\left(y'\right) + (1-\lambda)\phi\left(x'\right)\end{aligned}$$

for all $\lambda \in (0, 1)$, where the first line uses (23), the second line uses equation (22), the third line follows from algebraic manipulations, the fourth line uses (20) and (21), and the last line substitutes the definitions of x' and y'. Since this inequality holds for all $\lambda \in (0, 1)$, in particular, it must hold for γ and $1 - \gamma$:

$$\phi\left(\gamma y' + (1-\gamma)x'\right) < \gamma\phi\left(y'\right) + (1-\gamma)\phi\left(x'\right)$$

and

$$\phi\left(\gamma x' + (1-\gamma)y'\right) < \gamma\phi\left(x'\right) + (1-\gamma)\phi\left(y'\right).$$

Combining these two inequalities, gives

$$\phi\left(\gamma y' + (1-\gamma)x'\right) + \phi\left(\gamma x' + (1-\gamma)y'\right) < \phi\left(y'\right) + \phi\left(x'\right),$$

showing that Residual Risk Aversion fails. The proof for Residual Risk Seeking is analogous. □

C.6   Proof of Observations 2, 3, and 4

Observation 2 is due to Proposition 3 and the fact that KM coincides with EDU if and only if ϕ is affine. Observation 3 follows from Proposition 3 and the KM representation of EZ given in Example 4. Observation 4 follows directly from Proposition 3. ■

C.7 Proof of Proposition 4

Before presenting the proof, it is helpful to rewrite Stochastic Impatience in terms of the KM model. Stochastic Impatience holds if and only if, for all $x, y, c \in C$ with $x > y > c$ and all $t_1, t_2 \in \mathbb{N}$ with $t_1 < t_2$,

$$\phi \left(\frac{\sum_{\tilde{t} < t_1} D(\tilde{t}) u(c) + \sum_{\tilde{t} \geq t_1} D(\tilde{t}) u(x)}{\sum_{\tilde{t}=1}^{\infty} D(\tilde{t})} \right) + \phi \left(\frac{\sum_{\tilde{t} < t_2} D(\tilde{t}) u(c) + \sum_{\tilde{t} \geq t_2} D(\tilde{t}) u(y)}{\sum_{\tilde{t}=1}^{\infty} D(\tilde{t})} \right)$$

$$\geq \phi \left(\frac{\sum_{\tilde{t} < t_1} D(\tilde{t}) u(c) + \sum_{\tilde{t} \geq t_1} D(\tilde{t}) u(y)}{\sum_{\tilde{t}=1}^{\infty} D(\tilde{t})} \right) + \phi \left(\frac{\sum_{\tilde{t} < t_2} D(\tilde{t}) u(c) + \sum_{\tilde{t} \geq t_2} D(\tilde{t}) u(x)}{\sum_{\tilde{t}=1}^{\infty} D(\tilde{t})} \right)$$

For notational simplicity, let $u_x \equiv u(x) > u_y \equiv u(y) > u_c \equiv u(c) \geq 0$, and let $d_t \equiv \frac{\sum_{\tilde{t} \geq t} D(\tilde{t})}{\sum_{\tilde{t}=1}^{\infty} D(\tilde{t})} \in (0,1)$. Then, the previous condition becomes

$$\phi \left(d_{t_1} u_x + (1 - d_{t_1}) u_c \right) + \phi \left(d_{t_2} u_y + (1 - d_{t_2}) u_c \right)$$

$$\geq \phi \left(d_{t_1} u_y + (1 - d_{t_1}) u_c \right) + \phi \left(d_{t_2} u_x + (1 - d_{t_2}) u_c \right)$$

for all $u_x, u_y, u_c \in [0, u(\bar{x})]$ with $u_x > u_y > u_c$ and all $d_{t_1}, d_{t_2} \in \left\{ \frac{\sum_{\tilde{t} \geq t}^{\infty} D(\tilde{t})}{\sum_{\tilde{t}=1}^{\infty} D(\tilde{t})} \right\}_{t \in \mathbb{N}}$ with $d_{t_1} > d_{t_2}$.

Proof of Part (i).

Let $\phi = g \circ \ln$ for some increasing and convex function g. It suffices to show that

$$\begin{aligned} & g \left(\ln \left(\alpha u_x + (1 - \alpha) u_c \right) \right) + g \left(\ln \left(\gamma u_y + (1 - \gamma) u_c \right) \right) \\ & \geq g \left(\ln \left(\alpha u_y + (1 - \alpha) u_c \right) \right) + g \left(\ln \left(\gamma u_x + (1 - \gamma) u_c \right) \right) \end{aligned} \tag{24}$$

for all $u_x > u_y > u_c \geq 0$ and all $0 < \gamma < \alpha \leq 1$. Let

$$z \equiv \ln \left[\frac{\left(\alpha u_y + (1 - \alpha) u_c \right) \left(\gamma u_x + (1 - \gamma) u_c \right)}{\alpha u_x + (1 - \alpha) u_c} \right].$$

Algebraic manipulations establish that:

$$z < \min \left\{ \ln \left(\alpha u_y + (1 - \alpha) u_c \right), \ \ln \left(\gamma u_x + (1 - \gamma) u_c \right) \right\},$$

and

$$\ln \left(\alpha u_x + (1 - \alpha) u_c \right) + z = \ln \left(\alpha u_y + (1 - \alpha) u_c \right) + \ln \left(\gamma u_x + (1 - \gamma) u_c \right).$$

Therefore, by the convexity of g, we have

$$g \left(\ln \left(\alpha u_x + (1 - \alpha) u_c \right) \right) + g(z) \geq g \left(\ln \left(\alpha u_y + (1 - \alpha) u_c \right) \right) + g \left(\ln \left(\gamma u_x + (1 - \gamma) u_c \right) \right).$$

With some algebra, it can also be shown that

$$z < \ln \left(\gamma u_y + (1 - \gamma) u_c \right).$$

Then, (24) follows from the fact that g is increasing.

Proof of Part (ii).

Take $u_c = 0$, $t_1 = 1$, and let $\phi = g \circ \ln$ for some strictly concave function g. Then, Stochastic Impatience implies:

$$g\left(\ln u_x\right) - g\left(\ln\left(1 - d_t\right) + \ln u_x\right) \geq g\left(\ln u_y\right) - g\left(\ln\left(1 - d_t\right) + \ln u_y\right)$$

for $t = 2, 3, \dots$. But note that

$$\ln u_x - \left[\ln\left(1 - d_t\right) + \ln u_x\right] = \ln u_y - \left[\ln\left(1 - d_t\right) + \ln u_y\right] = -\ln\left(1 - d_t\right).$$

Then, it follows from $\ln u_x > \ln u_y$ and the concavity of g that

$$g\left(\ln u_x\right) - g\left(\ln\left(1 - d_t\right) + \ln u_x\right) < g\left(\ln u_y\right) - g\left(\ln\left(1 - d_t\right) + \ln u_y\right),$$

which shows that Stochastic Impatience fails.

Proof of Parts (iii) and (iv).

The claims in parts (iii) and (iv) follow directly from (i) and (ii). In particular, any preferences that admit a KM representation (ϕ, u, β) with $\phi(x) \equiv \frac{x^{1-\sigma}}{1-\sigma}$ for $\sigma > 1$ (so that ϕ is more concave than the log) violate Stochastic Impatience.

This concludes the proof of the Proposition. ∎

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