Life Pro Tip: How to Responsibly Obtain More Real Numbers?

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Prerequisites

ZFC

The mainstream set-theoretic axiomization of mathematics. Notice that we embrace **Axiom of Choice** in this talk. We also assume ZFC is consistent outside of ZFC.

Cardinality

Informally, cardinalities are equivalent classes on all sets with respect to the equivalent relation stating that there is a bijection between two sets. Under Choice, cardinalities are totally ordered by $\leq$ where $\alpha \leq \beta$ means there is a injection from α to β.

Plus the following:

- $\aleph_0$, the cardinality of $\mathbb{N}$.
- $\aleph_1$, the smallest cardinality that is bigger than $\aleph_0$.
- $\aleph_2$, the smallest cardinality that is bigger than $\aleph_1$.
- $2^{\aleph_0}$, the cardinality of $\mathbb{R}$, the *continuum*.

The Size of the Continuum

By Cantor's diagonlization proof, we have $\aleph_0 < 2^{\aleph 0}$ and thus

$$\aleph_0 < \aleph_1 \leq 2^{\aleph_0}$$

Continuum Hypothesis:

$$\aleph_1 = 2^{\aleph_0}$$

It seems to be naturally true as you cannot find a set with size strictly between $\mathbb{N}$ and $\mathbb{R}$. Nevertheless, mathematicians failed to prove it, including Cantor himself. It became the first among Hilbert's 23 problems.

Gödel's Incompleteness

In the last century, people gradually realize that the continuum hypothesis is unsolvable/independent within ZFC.

Given a set of formulas Γ, we use **Con(Γ)** to denote a formula which asserts consistency of Γ, i.e, no contradiction is derivable from Γ.

Second Incompleteness Theorem [Gödel, 1930] (ZFC)

Assuming ZFC is consistent, neither Con(ZFC) nor $\neg$Con(ZFC) is provable.

This indicates ZFC is an incomplete axiom system in the sense that it cannot decide truthhood of all statements.

Relative Consistency of ZFC+CH

Gödel showed a weaker (in comparison to directly proving CH from ZFC) but still positive result:

Relative Consistency of ZFC+CH [Gödel, 1938] (ZFC)

Con(ZFC) $\implies$ Con(ZFC+CH).

Assuming *metamathematically* that ZFC is consistent, this will imply that there is no proof of $\neg$CH from ZFC: if ZFC proves $\neg$CH, then you would derive a contradiction from ZFC+CH. The above $\implies$ will transfer the contradiction to ZFC itself, which is prohibited by the initial assumption.

We then say ZFC+CH is **relatively consistent** to ZFC, i.e., ZFC+CH is as consistent as ZFC itself. We are not risking anything by assuming CH in addition to ZFC.

Gödel uses inner models (constructible universe) to prove this result.

Relative Consistency of ZFC+$\neg$CH

Around thirty years later, Paul Cohen invented a method called **forcing** to show the other way. He then won the Fields medal in 1966.

Relative Consistency of ZFC + $\neg$CH [Cohen, 1963] (ZFC)

Con(ZFC) $\implies$ Con(ZFC+$\neg$CH).

Thus, ZFC + $\neg$CH is also as consistent as ZFC itself.

In particular, CH is **independent** of ZFC, i.e., neither CH nor $\neg$CH is provable from ZFC.

The End.

Wait A Minute...

Independence is an unsatisfactory answer, especially if you are a Platonist. People are separated into two schools of thoughts, one accepting ZFC+CH, one accepting ZFC $+ \neg$CH, and Oualid who is unfortunately a constructivist.

To the majority of set theorists, CH is not desirable either because it is too restrictive or it brings pathological consequences. In addition, the idea of $V = L$ (constructible universe) that is used to prove relative consistency of ZFC+CH gets rejected almost unanimously.

One of such examples is Freiling's axiom of symmetry, which is equivalent to $\neg$CH. Penelope Maddy wrote an excellent philosophical paper summarizing major arguments in this unsettled debate, called *Believing the Axioms*.

Negation of CH

From now on, we will focus on the negation of CH, which says $2^{\aleph_0} \geq \aleph_2$. But what is the size of the continuum exactly?

Cohen's original forcing that opens the possibility of violating CH can be used to show for any $\kappa > \aleph_0$ with uncountable cofinality (at least $\aleph_n$ for $n \geq 2$),

$$\text{Con(ZFC)} \implies \text{Con}(\text{ZFC} + (2^{\aleph_0} = \kappa))$$

In fact, one can argue further to show that one can violate generalized continuum hypothesis (GCH) in pretty much any way they want. Therefore, this provides no positive tendency to assign $2^{\aleph_0}$ to any particular cardinality. We need a different path/principle and we will talk about forcing axioms.

Forcing

Let us explicitly assume ZFC is consistent. We then have access to a universe of set theory, called V. This implicitly violates Gödel's incompleteness as our meta-theory is ZFC, which forces V to be a set and leads to prove its own consistency. Fortunately, one can translate this informal and inaccurate model theory nonsense to purely syntactical proofs.

What Gödel did is essentially to construct an inner model $L \subseteq V$, which provably remains to be a model of ZFC. This procedure provides a proof of Con(ZFC) $\implies$ Con(ZFC$+\phi$) for whatever extra property ϕ of L. For Cohen's forcing argument, however, we need to go beyond V.

We start with a partially ordered set $\mathbb{P}$ in V. We need two definitions.

Dense subsets and filters

$D \subseteq \mathbb{P}$ is dense if for all $p \in \mathbb{P}$, there exists $q \in D$ such that $q \leq p$.
$G \subseteq \mathbb{P}$ is a filter if it is nonempty, closed upwards, and contains a common lower bound for each pair of elements in G.

Forcing Continued

Given $\mathfrak{D}$ a collection of dense subsets in $\mathbb{P}$, a filter G in $\mathbb{P}$ is $\mathfrak{D}$-generic if $G \cap D \neq \emptyset$ for all $D \in \mathfrak{D}$. For most posets $\mathbb{P}$ of interest, generic filters generally will not exist in V. However, G could potentially be added into V by building up a new model $V[G]$ of ZFC, "generated" by V and G. The elements of V will be reconstructed back in $V[G]$; we use genericity of G to recover various properties of G as they will be encoded by the choice of $\mathfrak{D}$.

One can loosely picture this forcing extension as a field extension where V corresponds to a field, G is an imaginary number, and various dense sets form a polynomial approximation of the imaginary number.

For instance, Cohen's forcing is a poset of all functions $\aleph_2 \times \aleph_0 \to \{0, 1\}$ with finite supports under reverse inclusion. Then $\bigcup G$ becomes a function from $\aleph_2 \times \aleph_0$ to $\{0, 1\}$, which can be curried into a map $\aleph_2 \to 2^{\aleph_0}$ and it is actually an injection using density argument. This shows $2^{\aleph_0} \geq \aleph_2$ in $V[G]$ by witness of G.

Forcing Continued Continued

There is a problem in the argument above: $2_0^{\aleph}$ and $\aleph_2$ are defined in V and G does witness $2^{\aleph_0} \geq \aleph_2$ of V; how do we know G say anything about $2_0^{\aleph}$ and $\aleph_2$ of $V[G]$ as sets in V are reassembled in $V[G]$?

This is resolved by noticing Cohen's poset $\mathbb{P}$ satisfies c.c.c./countable antichain condition, i.e., any antichain (subset where any two elements are incomparable) of $\mathbb{P}$ is at most countable. It is a combinatorial fact that any forcing using a c.c.c. poset will preserve cardinals and we are good to go.

Forcing provides an excellent tool to mass produce independence results as you can design various posets with desired generic filters and force with it.

Martin's Axiom

Let's return back to our search for good axiom candidates. One plausible philosophical principle is to maximize our true universe, i.e., our universe is large enough that it simultaneously contains everything that could potentially exist. In language of forcing, it says that any reasonable poset has a generic filter existing in V. Any such principle is a **forcing axiom**.

For each cardinal $\aleph_0 < \kappa < 2^{\aleph_0}$, MA_κ is the following forcing axiom:

Axiom: MA_κ

For any partially order set $(\mathbb{P}, \leq)$ with c.c.c. and a collection $\mathfrak{D}$ of dense subsets of $\mathbb{P}$ with $|\mathfrak{D}| \leq \kappa$, there exists a $\mathfrak{D}$-generic filter G.

We define **Martin's axiom** to be the statement

$$\forall \kappa (\aleph_0 < \kappa < 2^{\aleph_0} \implies \mathsf{MA}_\kappa)$$

For the purpose of our talk, we will focus on $\mathsf{MA}_{\aleph_1}$.

Independence of Martin's Axiom

Note that $\mathsf{MA}_{\aleph_0}$ is actually provably true in ZFC (it is essentially Baire Category theorem) while $\mathsf{MA}_{2^{\aleph_0}}$ is provably false in ZFC. Therefore, we immediately have the following theorem in ZFC:

$$\mathsf{MA}_{\aleph_1} \implies 2^{\aleph_0} \neq \aleph_1 \implies 2^{\aleph_0} \geq \aleph_2$$

We have that $\mathsf{MA}_{\aleph_1}$ is indeed a strengthening of $\neg$CH, which gives relative consistency of $\mathsf{ZFC} + \neg\mathsf{MA}_{\aleph_1}$ via Con(ZFC+CH):

Relative Consistency of ZFC+$\neg\mathsf{MA}_{\aleph_1}$ (ZFC)

$\mathsf{Con}(\mathsf{ZFC}) \implies \mathsf{Con}(\mathsf{ZFC} + \neg\mathsf{MA}_{\aleph_1})$

With a bit of extra work (iterated forcing, in comparison to a tower of field extensions), we can also show:

Relative Consistency of ZFC+$\mathsf{MA}_{\aleph_1}$ (ZFC)

$\mathsf{Con}(\mathsf{ZFC}) \implies \mathsf{Con}(\mathsf{ZFC} + \mathsf{MA}_{\aleph_1})$

Martin's Maximum

Martin's axiom is not strong enough to decide the size of the continuum,. One can show relative consistency of ZFC $+ \mathsf{MA}_{\aleph_1} + (2^{\aleph}_0 = \kappa)$ for any reasonable $\kappa \geq \aleph_2$.

Foreman-Magidor-Shelah strengthen $\mathsf{MA}_{\aleph_1}$ by adding more posets into the statement; we relax the requirement of c.c.c. by something named stationary-preserving and have the following forcing axioms

Axiom: MM (note that we drop the $\aleph_1$ subscript)

For any partially order set $(\mathbb{P}, \leq)$ which is stationary-preserving and a collection $\mathfrak{D}$ of dense subsets of $\mathbb{P}$ with $|\mathfrak{D}| \leq \aleph_1$, there exists a $\mathfrak{D}$-generic filter G.

$\mathbb{P}$ is stationary-preserving means whenever G is a generic filter, $V[G]$ preserves all stationary subsets of $\aleph_1$ in V. Stationary subset of $\aleph_1$ is a set intersecting all closed unbounded subsets (clubs) of $\aleph_1$. Clubs are treated as positive-measure sets and non-stationary sets as zero-measure sets. We have a filter of clubs and its dual ideal of non-stationary sets.

Consequences of MM

By the fact that any c.c.c. poset is also stationary-preserving, we do have MM $\implies$ MA$_{\aleph_1}$. If we further relax the stationary-preserving requirement, the stronger axiom will become provably inconsistent with ZFC. Hence, MM is a maximal extension of MA$_{\aleph_1}$, which is suggested by its name.

Unlike MA$_{\aleph_1}$, MM is strong enough to resolve CH:

(ZFC)

MM $\implies (2^{\aleph_0} = \aleph_2)$

It pins the size of continuum to exactly $\aleph_2$, the smallest possible value after negating CH. MM also determines many interesting behaviors of $\aleph_1$, such as saturation of non-stationary ideals on $\aleph_1$.

Independence of MM?

As MM $\implies$ $\mathrm{MA}_{\aleph_1}$ $\implies$ $\neg$CH, we have a similar result:

Relative Consistency of ZFC+¬MM (ZFC)

$\mathrm{Con}(\mathrm{ZFC}) \implies \mathrm{Con}(\mathrm{ZFC} + \neg \mathrm{MM})$

However, assuming ZFC is consistent, it is actually provable that the other case is not provable in ZFC:

$\mathrm{Con}(\mathrm{ZFC}) \not\Longrightarrow \mathrm{Con}(\mathrm{ZFC} + \mathrm{MM})$

This is because MM is also strong enough to imply the existence of a large cardinal. Suppose the above implication is a theorem of ZFC. Assuming ZFC is consistent, we also have consistency of ZFC+IC via MM (IC stands for the existence of an inaccessible cardinal).

Problem with Large Cardinals

Now imagine our true universe V satisfies ZFC+IC. Within V, we can build a new universe V' by extracting information from an inaccessible cardinal κ at hand. This V' is a set as our meta-theory is ZFC and hence we prove consistency of ZFC within ZFC, which directly violates Gödel incompleteness theorem. Thus, we proved that we cannot give a proof of relative consistency of ZFC+MM, assuming ZFC is consistent.

It is bad news as we cannot guarantee no new contradiction will arise and adding MM as a new axiom does hurt consistency of our axiomatic system, unlike others. There is even a possibility that someone might at some point come up with a proof of inconsistency of ZFC+MM from ZFC, assuming consistency of ZFC. Then why are there people promoting this axiom?

Benefits with Large Cardinals

Even though we cannot prove relative consistency of ZFC+MM from ZFC, we can prove it with some extra assumption:

Relative Consistency of ZFC+MM (ZFC+SCC)

$\mathrm{Con}(\mathrm{ZFC}) \implies \mathrm{Con}(\mathrm{ZFC}+\mathrm{MM})$

SCC is an axiom asserting the existence of a supercompact cardinal, which is an extremely large cardinal even to the standard of large cardinals, the smallest of which is already enough to prove consistency of ZFC itself.

Therefore, the question whether one should believe MM is consistent with ZFC (dropping relativity intentionally) gets deferred to the question whether one should believe free usage of large cardinals.

New Results

Hugh Woodin, one of the greatest of set theorists at our time, has a completely different approach and conjectures a different axiom called $(*)$, which primarily works towards the direction of L and axiom of determinacy. Woodin's axiom has a lot of interesting consequences and also pins $2^{\aleph_0}$ to $\aleph_2$ for completely different reasons ($\aleph_2$ behaves differently for many set-theoretic properties from cardinalities beyond it). A surprising result is proved in 2021 where MM^{++} is a variation of MM:

[Asperó & Schindler 2021] (ZFC)

$\text{MM}^{++} \implies (*)$

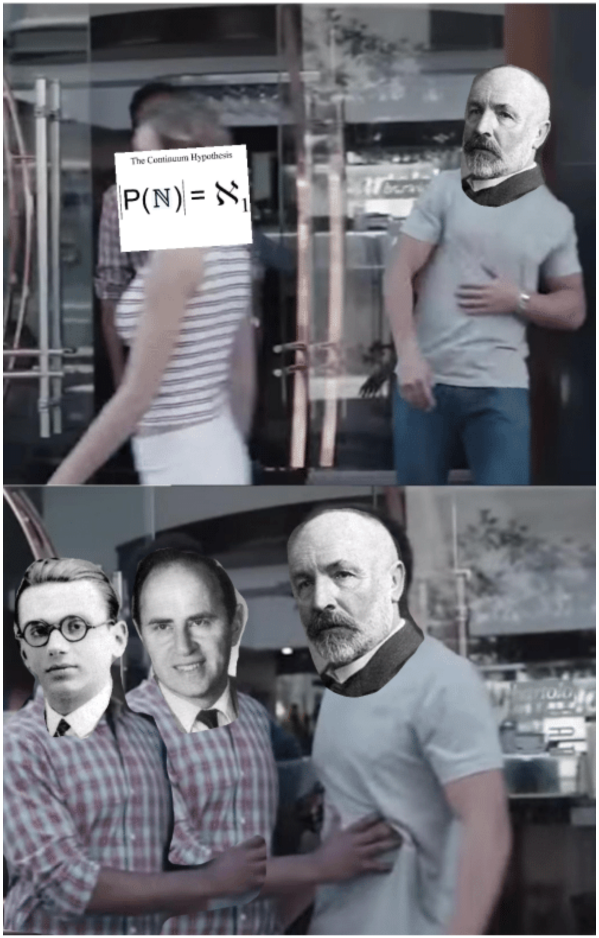
This adds much more weights to credibility of Martin's Maximum, even though it probably still far from being accepted by general mathematical audience.

The End

So far, the two predominant proposals regarding the size of continuum are $\aleph_1$ and $\aleph_2$. Kurt Gödel himself believed that $2^{\aleph_0} = \aleph_2$ at some point even before the birth of forcing. As Gödel is a god, you should know who to follow regarding the question about CH.

Thanks and enjoy pizzas!

Meme



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