Pontryagin Duality on LCA Groups

Darren Zheng

December 2023

Abstract

Both the discrete and real Fourier transforms have many applications, drawing from a rich theory. In this write-up, we will discuss a generalization of the Fourier transform onto locally compact abelian groups, introducing techniques in abstract harmonic analysis. We begin by talking about what it means to integrate over a group before introducing the Haar measure. We will touch on Pontryagin duality and finish by discussing some consequences of this result.

Contents

1 LCA Groups

We say a group is topological group if both the group operation $x \cdot y \to xy$ and inverse map $x \to x^{-1}$ are continuous maps. Furthermore, we say G is locally compact, if for every point $x \in G$, there exists a compact neighborhood containing G and G is Hausdorff. We will mostly be dealing with locally-compact abelian groups, which we shorten as LCA groups. Being a LCA group already gives us a few basic properties following directly from the aforementioned continuity.

Proposition 1. *Let G be a topological group. Then,*

(a) If $U \subset G$ is open, then xU, Ux, and U^{-1} is also open for any $x \in G$. We define $xU = \{xy \mid y \in U\}$ and $U^{-1} = \{y^{-1} \mid y \in U\}$.

(b) If H is a subgroup of G, the closure of H is a subgroup.

(c) If H is a open subgroup of G, then H is closed.

(d) If A, B are compact (or open) sets in G, then $AB = \{xy \mid x \in A, y \in B\}$ is compact (or open).

(e) A set $U \subset G$ is compact (or open) if and only if there exists some $x \in U$ and compact (or open) set E containing the group identity e, such that $xE = U$.

For our future discussion on the Fourier transform, we need to define a suitable notion of integration on LCA groups. While the normal Riemann integral is usually defined in terms of upper and lower Riemann sums, we present an alternative interpretation. Define for each $n \in \mathbb{N}$, $\mathbf{1}_n(x) = \chi_{[-\frac{1}{2n}, \frac{1}{2n}]}(x)$, an indicator function on the interval $[-\frac{1}{2n}, \frac{1}{2n}]$. Then, we can define

$$(f : \mathbf{1}_n) := \inf\{\sum_{i=1}^{m} c_i \mid c_i, \ldots, c_m > 0, \exists x_1, \ldots, x_m \in \mathbb{R} \text{ s.t. } f(x) \leq \sum_{i=1}^{m} c_i \mathbf{1}_n(x_i + x)\}$$

Intuitively, the above definition is redefining the partition as a sequence x_i are left and right translations the indicator functions, while the c_i approximate the value on each partition. As $n \to \infty$, we are considering the linear combination on a finer partition. There is a $\frac{1}{n}$ factor that appears due to the length of the interval. Thus, the Riemann integral can be written as

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{n\to\infty} \frac{(f : \mathbf{1}_n)}{n} = \lim_{n\to\infty} \frac{(f : \mathbf{1}_n)}{(\chi_{[0,1]} : \mathbf{1}_n)}$$

For a general LCA group G, we can redefine this using neighborhoods. Since our space is locally compact, fix a compact neighborhood K of $e \in G$ and let $f_0 \geq 0$ be any function not identically zero. Then, let

$$\int_G f(x)\,dx = \lim_{K\to\{e\}} \frac{(f : \mathbf{1}_U)}{(f_0 : \mathbf{1}_U)}$$

Again, we see the same idea of approximation by step functions, where $x_1, \ldots, x_m$ now attempt to translate U on the left to cover K. It is dependent on the choice of function f_0 and compact set K (if there are multiple) to get an exact value. This integral is actually called the Haar integral. In the construction above, the last step is to prove that the limit actually exists.

It is worth noting that normally in measure theory, one constructs a measure first to then induce an integral. If one were to do that in this case, they could consider for $A, K, U \subseteq G$, where K compact and U open,

$$\mu(A) = \lim_{U\to\{e\}} \frac{\inf\{n \mid \exists x_1, \ldots, x_m \text{ s.t. } A \subseteq \cup_{i=1}^n x_i U\}}{\inf\{n \mid \exists x_1, \ldots, x_m \text{ s.t. } K \subseteq \cup_{i=1}^n x_i U\}}$$

In practice, this expression is more difficult to work with, so our proof will investigate the linear functional to get the existence of the Haar measure using the Riesz representation theorem.

2 Haar Measure

Let X be a set and Σ a σ-algebra over X. We call a function μ from $\Sigma \to [0, \infty)$ a measure if it satisfies two properties

1. $\mu(\emptyset) = 0$.
2. For all $E \in \Sigma$, $\mu(E) \geq 0$.
3. $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$, where E_i are all disjoint

Roughly speaking, a measure assigns a notion of size to subsets of a set. The reader may be familiar with the Lebesgue measure on $(\mathbb{R}, +)$ as a topological group, which remains invariant under left translation ($\mu([a, b]) = \mu([a + c, b + c])$). A left Haar measure can be viewed as an analogue of this which preserves the size under a left group action ($\mu([a, b]) = \mu([ca, cb])$). For our purposes, it suffices to have our Σ be the Borel σ-algebra generated by countable unions and intersections of closed and open sets of our LCA group G.

Definition 1. A Radon measure $\mu : \Sigma \to [0, \infty)$ is a measure on a Borel σ-algebra Σ, such that:

1. (inner regular) $\mu(E) = \sup\{\mu(K) \mid K \text{ compact }, K \subset E\}$
2. (outer regular) $\mu(E) = \inf\{\mu(U) \mid U \text{ open }, E \subset U\}$
3. (finite on compact sets) $\mu(K) < \infty$ for any compact $K \in \Sigma$

Definition 2. A Haar measure refers to any Radon measure such that $\mu(gE) = \mu(E)$ for all $g \in G$ and $E \in \Sigma$.

Theorem 2.1 (Existence and Uniqueness of Haar measure). *Let G be any LCA group. Then, G admits a Haar measure μ. Furthermore, μ is unique up to scaling by some real positive number.*

Proof. As aforementioned, we want to construct a linear functional L on $C_c(G)$ (continuous functions of compact support on G), which will then induce a measure via Riesz representation. This linear functional is precisely $L_{\mathbf{1}_U}(f) = (f : \mathbf{1}_U)$, where $\mathbf{1}_U$ is just an indicator function on U. It is not hard to check that $L_{\mathbf{1}_U}$ is:

(a) left-invariant ($(f(x) : \mathbf{1}_U) = (f(x + g) : \mathbf{1}_U)$ for any $g \in G$)

(b) subadditive ($(f_1 + f_2 : \mathbf{1}_U) \leq (f_1 : \mathbf{1}_U) + (f_2 : \mathbf{1}_U)$)

(c) monotone ($f_1 \leq f_2 \implies (f_1 : \mathbf{1}_U) \leq (f_2 : \mathbf{1}_U)$)

(d) constants factor out ($(cf : \mathbf{1}_U) = c(f : \mathbf{1}_U)$)

The issue is that we need the subadditivity to be additivity. Fortunately, this is not far from the truth. We need to prove the following lemma.

Lemma 1. *If $f_1, f_2 \in C_c(G)$, then for any $\epsilon > 0$, there is a neighborhood V containing the identity, such that $(f_1 : \mathbf{1}_U) + (f_2 : \mathbf{1}_U) \leq (f_1 + f_2 : \mathbf{1}_U) + \epsilon$ so long as supp$(\mathbf{1}_U) \subset V$.*

For full details on existence, check Theorem 2.10 in Folland. □

Due to the uniqueness up to scaling, we can choose normalize the measure, such that $\mu(G) = 1$; thus, we can speak of the Haar measure for LCA groups.

As an aside, there is a completely analogous story for right Haar measures, which we will not cover. Although, in general, the left and right Haar measures are not the same, they do coincide in the case of abelian groups.

3 Dual Groups

For any abelian group A, we define a character χ to be a homomorphism from G to $\mathbb{T} = \{x \in \mathbb{C} \mid ||x|| = 1\}$, the multiplicative group of complex numbers. We define the set $\widehat{A}$ to contain all characters on A.

Proposition 2. *The pointwise product of characters turns $\widehat{A}$ into an abelian group.*

Proof. For any two characters $\chi, \nu \in \widehat{A}$ and $a, b \in A$, we have

$$\chi\nu(ab) = \chi(ab)\nu(ab) = \chi(a)\chi(b)\nu(a)\nu(b) = \chi(a)\nu(a)\chi(b)\nu(b) = \chi\nu(a)\chi\nu(b)$$

It is clear to see that the pointwise product is abelian. Next, the identity element is the trivial map $\mathbf{1}$ sending all elements to $1 \in \mathbb{T}$. For $a, b \in A$,

$$\mathbf{1}(ab) = 1$$
$$\mathbf{1}(a)\mathbf{b} = 1 \cdot 1 = 1$$

This means $\mathbf{1} \in \widehat{A}$. Additionally,

$$\chi\mathbf{1}(a) = \chi(a)\mathbf{1}(a) = \chi(a)1 = \chi(a)$$

Finally, we need to check that we have the necessary inverses. For $\chi \in \widehat{A}$, define $\chi^{-1}(a) = \frac{1}{\chi(a)}$. Then, we need to check $\chi^{-1}(a) \in \widehat{A}$.

$$\chi^{-1}(ab) = \frac{1}{\chi(ab)} = \frac{1}{\chi(a)\chi(b)} = \chi^{-1}(a)\chi^{-1}(b)$$

Additionally,

$$\chi(a)\chi^{-1}(a) = \frac{\chi(a)}{\chi(a)} = 1 = \mathbf{1}(a)$$

Thus, we've verified that $\widehat{A}$ is an abelian group. □

We call this $\widehat{A}$ the dual group, or Pontryagin dual, of A. It is clear that $\widehat{A}$ is a subset of all continuous functions from G to $\mathbb{T}$. We can equip $\widehat{A}$ with the topology of uniform convergence on compact sets, or compact-open topology. For any compact $K \subseteq A$ and open $U \in\subseteq \mathbb{T}$, define the set $V(K, U)$ to contain continuous maps $f : A \to \mathbb{T}$, such that $f(K) \subseteq U$. This forms a basis for the compact-open topology. Surprisingly, $\widehat{A}$ remains a LCA group.

Theorem 3.1. *If A is a LCA group, then $\widehat{A}$ is also a LCA group with the compact-open topology.*

Proof. To check that groups operations (written additively) is continuous, it is enough to check that the map $\phi : \hat{A} \times \hat{A} \to \hat{A}$ sending $\phi(\chi, \nu) = \nu - \chi$ is continuous (by fixing $\nu = \mathbf{1}$, we'd get continuity of the inverse map; then, the product map is just a composition of ϕ and the inverse map). Fix some $\epsilon > 0$. We already know that χ, ν are continuous having belonged to $\hat{A}$. Suppose that there are sequences of characters $\{\chi_i\}_{i=1}^{\infty}, \{\nu_i\}_{i=1}^{\infty}$ that converge to χ, ν respectively. We need to show that $\{\nu_i - \chi_i\}_{i=1}^{\infty} \to \nu - \chi$. From continuity, we can establish for sufficiently large n,

$$|\chi - \chi_i| < \frac{\epsilon}{2} \text{ and } |-\nu + \nu_i| < \frac{\epsilon}{2}$$

Using the triangle inequality,

$$|\chi - \chi_i - (\nu - \nu_i)| < |\chi - \chi_i| + |\nu - \nu_i| < \epsilon$$

Now, we need to check that the dual group is locally compact. We only present a sketch. Folland proves this in Theorem 4.2 by showing $\widehat{A}$ is exactly the spectrum of $L^1(A)$. Afterwards, since $\widehat{A}$ is a subset of $L^\infty(A)$ equipped with the weak$*$ topology, the compact open and weak$*$ topologies coincide. Under the latter topology, $\widehat{A}$ lies on the closed unit ball in $L^\infty(A)$. By Alaoglu's theorem, this ball is compact, so we can conclude $\widehat{A}$ is locally compact. □

4 Duality Theorems

Now, that we've set up much of the definition, we can look at two strong results about the structure of LCA groups.

Theorem 4.1 (Pontryagin Duality). *Let G denote a LCA group. Then, there is an isomorphism ϕ between G and $\widehat{\widehat{G}}$ given by*

$$\phi(g) = \delta_g$$

, where $\delta_g(\chi) = \chi(g)$, so it is the evaluation map.

Proof. The complete proof is too long to include in entirety. It is suggested to look at Theorem 4.31 in Folland. We will sketch very briefly some details.

It is relatively clear that ϕ is a homomorphism. For all $a, b \in G$ and $\chi \in \widehat{G}$,

$$\phi(ab)(\chi) = \delta_{ab}(\chi) = \chi(ab) = \chi(a)\chi(b) = \delta_a(\chi)\delta_b(\chi) = [\phi(a)(\chi)][\phi(b)(\chi)]$$

The fact that ϕ is injective follows from the Gelfand-Raikov theorem that states characters separate points. More precisely, for any a, b, there exists a character ν, such that $\nu(a) \neq \nu(b)$. If $\phi(a) = \phi(b)$, this relationship must suggest $\chi(a) = \chi(b)$ for all $\chi \in \widehat{A}$. Via the aforementioned Gelfand-Raikov theorem, this means $a = b$.

To prove surjectivity between topological groups, it suffices to show that the image of the homomorphism is dense and closed. In our case, we want to show that $\phi(G)$ is dense and closed in $\widehat{\widehat{G}}$. To verify that the map is dense, suppose for the sake of contradiction there is some subset $U \subset \widehat{\widehat{G}}$ that does not intersect the image. Then, one can construct a non-zero function f, whose support lies in U, alongside a sequence of functions f_n, such that $\widehat{f_n} \to f$. By Fourier inversion, we can get that $f_n \to \widehat{f}$. By construction, $f_n = 0$ on $\phi(G)$. However, this is a contradiction as $f_n \to \widehat{f}$ means $f = 0$, when we explicitly said it was non-zero.

For closed, we can instead show that ϕ is a proper map (preimages of compact sets are compact), which, combined with the fact it is continuous, would mean it is closed. Details are omitted. $\square$

While a LCA group is not isomorphic to its dual in general, it is always isomorphic to the double dual. This theorem should remind the reader of a similar result in the case of vector spaces. For finite dimensional vectors spaces V, there is a canonical isomorphism into $\widehat{\widehat{A}}$, the double dual of V. Thus, we can view the locally compact conditions as a similar "finite-ness" restriction. With the double dual isomorphic to the original group, properties from one space can be used to conclude properties of the dual. A topological group is compact if every open cover admits a finite subcover. A topological group is discrete if every single element set is open.

Theorem 4.2. *If a LCA group G is compact, then $\widehat{G}$ is discrete. If G is discrete, then $\widehat{G}$ is compact.*

Proof. Recall for topological groups, being open is preserved under left translations, so it suffices to check at the identity. If G is compact, then the set $\mathrm{Hom}(G, S)$ of homomorphisms from G into any open subset S of $\mathbb{T}$ is itself an open set in $\widehat{G}$. However, the image of a homomorphism in $\mathrm{Hom}(G, S)$ must be a subgroup of $\mathbb{T}$. But the only proper subgroup of $\mathbb{T}$ is the trivial group. Thus, the trivial character in $\mathrm{Hom}(G, S)$ must be open as a singleton set, so $\widehat{G}$ is discrete.

If G is discrete, then consider the indicator function of the identity $e \in G$

$$\chi_1(x) = \begin{cases} 1 & x = e \\ 0 & \text{otherwise} \end{cases}$$

Then, $\chi_1(x)$ is continuous from the discrete topology, so we know that $\chi_1(x)$ belongs to $L^1(G)$. In particular, this means that $L^1(G)$ has a multiplicative identity element. It is a result from functional analysis that this implies the spectrum of $L^1(G)$ is compact. From the same association as the proof of Theorem 3.1, the spectrum of $L^1(G)$ is $\widehat{G}$, so we are done. $\square$

5 Applications

The duality theorems suggest some very strong structures, the beginnings of abstract harmonic analysis. In particular, Pontryagin duality show many properties of the Fourier transform on $\mathbb{R}$ are due to it being a LCA group, rather than specifically $\mathbb{R}$. For example, both Plancherel's theorem and Fourier Inversion are consequences. Before stating the theorems, first, we remind ourselves of some definitions.

Definition 3 (Fourier Transform). Given a LCA group G and a Haar measure μ on G, the Fourier transform of a function $f : G \to \mathbb{C}$ is $\widehat{f} : \widehat{G} \to \mathbb{C}$, where

$$\widehat{f}(\chi) = \int_G F\overline{\chi}\, d\mu$$

Above, $\overline{\chi}$ is denoting the complex conjugate.

Note that $\widehat{\mathbb{R}} = \mathbb{R}$ and the characters of $\mathbb{R}$ are complex exponentials $e^{2\pi i n x}$ for some $n \in \mathbb{N}$, so we can recover the familiar Fourier transform $\widehat{f}(n) = \int_{-\infty}^{\infty} e^{-2\pi i n x} f(x)\, dx$. For more examples, we note that $\widehat{\mathbb{Z}/n\mathbb{Z}} = \mathbb{Z}/n\mathbb{Z}$ and $\widehat{Z} = \mathbb{T}$.

Denote the space $L^p(G)$ as the space of continuous p-integrable functions on G, such that $\int_G |f|^p\, d\mu < \infty$, where $d\mu$ is a Haar measure. By convention, the case $p = 2$ is called the space of square integrable functions on G. We equip $L^p(G)$ with the standard p-norm.

Theorem 5.1 (Plancherel's Theorem). *There is a unique Haar measure given LCA group G, such that for any $f \in L^1(G)$,*

$$||f||_2 = \left|\left|\widehat{f}\right|\right|_2$$

, meaning its Fourier transform $\widehat{f} \in L^2(\widehat{G})$. In fact, this means the Fourier transform as a map from $L^1(G)$ to $L^2(\widehat{G})$ extends to an isomorphism of the completions $L^2(G)$ to $L^2(\widehat{G})$.

Above is the statement in full generality, but in the case of $G = \mathbb{R}$, we can see that it is the more familiar result that, assuming both $f, \widehat{f}$ are continuous and improperly Riemann integrable,

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} \left|\widehat{f}(x)\right|^2\, dx$$

For another example,

Theorem 5.2 (Fourier Inversion). *For any $f \in L^1(G)$,*

$$f(a) = \widehat{\widehat{f}}(\delta_{a^{-1}})$$

This means that the original function f can be recovered from its Fourier transform.

In the case of $G = \mathbb{R}$, we have that $f(a) = \int_{-\infty}^{\infty} \widehat{f}(x) e^{2\pi i a x}\, dx$, as $a^{-1} = -a$ in this case.

As a final example, with a bit of a categorical argument, one can use the observations above to prove a structure theorem to attempt to classify LCA groups.

Theorem 5.3 (Principal Structure Theorem for LCA groups). *If G is a LCA group, then G has an open subgroup H, such that $H \cong \mathbb{R}^n \times K$, where n is finite and K is a compact abelian group. Furthermore, if G is connected, then $G \cong \mathbb{R}^n \times K$, where n is finite and K a connected, compact abelian group.*

For the sake os space, a few of the proofs in this write-up were abbreviated. The interested reader is encouraged to check out any of the references for further details, applications, and perspectives.

Acknowledgements. I want to thank my mentor, Travis Leadbetter, for his always-present encouragement and willingness to answer questions. Additionally, I want to thank the UPenn Math directed reading program, especially its organizers Maxine Calle and Leo Guilhoto.

References

[1] Anton Deitmar, *A First Course in Harmonic Analysis*, https://doi.org/10.1007/978-1-4757-3834-6

[2] Gerald B. Folland, *A Course in Abstract Harmonic Analysis*, https://doi.org/10.1201/b19172

[3] Joe Diestel, Angela Spalsbury, *The Joys of Haar Measure* https://doi.org/10.1090/gsm/150