Minimal Surfaces and Plateau's Problem

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1 Introduction

Plateau's problem is not a single specific problem, but instead a set of problems that are aimed at solving boundary value minimal surface problems. A basic example of plateau's problem is the following: Given a boundary in $\mathbb{R}^3$, what is the minimal surface that corresponds to that boundary? There are numerous generalizations to this problem, such as ones in higher dimensional spaces as well as boundary problems where the boundary isn't even fixed. There are also generalizations that deal with minimal sets and graphs. The historical motivation for this problem deals with soap films where a wire boundary was dipped into a soapy solution and when removed slowly, would form a surface in the boundary.

2 Minimal Surfaces

A minimal surface in $\mathbb{R}^3$ is a surface that is locally area minimizing at each point on the surface with given boundary. More specifically, let Σ be a minimal surface in $\mathbb{R}^3$. Then Σ is minimal if every point in Σ has an ϵ -neighborhood U which has least area among all surfaces $S \subset \mathbb{R}^3$ with boundary ∂U. One thing to notice is that minimal surfaces need not have the smallest area amonst all surfaces with that boundary. For example, if we have a boundary which is two discs on top of one another by a fixed distance, we get two minimal surfaces depending where we decide to start the surface. For example, our minimal surface can fill out both of our disks and both disks won't be connect. Another minimal surface is when the disks aren't filled, but a surface connects the two disks.

For low dimensions, like $\mathbb{R}^2$, it is simple to come up with a minimal surface equation (an equation the minimizes our surface along our boundary). For example, in $\mathbb{R}^2$, our minimal surface equation is

$$\text{div}\frac{\nabla(f)}{(1+|\nabla f|^2)^{\frac{1}{2}}} = 0$$

.

Theorem 2.1 (Bernstein's Theorem) *Any solution to the minimal surface equation above on* $\mathbb{R}^2$ *is linear.*

Theorem 2.2 (Simons) *Bernstein's theorem above holds for minimal graphs where dimension* $n \leq 7$*.*

3 Orientable Generalized Surfaces

To generalize from simple examples of wireframes and soap films, one must ask whether topological type matters when calculating minimal surfaces. So far, all calculations require us to take genus of the surface into account. One way to perform this generalization is through the lense of integral currents.

Definition 3.1 (Current) *A current is a linear functional on a smooth differential form* $\Omega_c^m(M)$ *is the space of smooth m-forms on a smooth manifold. Our current would be the map:*

$$T : \Omega_c^m(M) \to \mathbb{R}$$

In a sense, currents act as integration on submanifolds, which is useful since we are trying to minimize some sort of area/volume in Plateau's problem.

Definition 3.2 (Integral Current) *A linear functional on the space of compactly supported differential forms that satisfies linearity, additivity, and generalized Stoke's.*

Integral currents generalize currents to nicer surfaces. For example, these surfaces end up being orientable, finite area and have an oriented boundary. Another important fact is that there exists an integral current with minimal mass among all integral currents with the same boundary. One can also look at integral currents as a generalization of Euler-Lagrange equations which are used to find local extrema in calculus of variation problems.

When finding minimal surfaces, it is important to note that "weird" behavior can occur with our given minimal surface. For example, we can run into a singularity on our surface which is simply a point on the minimal surface where we fail to be smooth or exhibit odd behavior such as an infinite convergence to a point that never reaches the point. One question that arises from this is the following: is it possible to admit interior singularities for our minimized surfaces? Simons answers this through his finding of the Simons cone: $x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2$. This object has an interior singularity at the origin. Consequently, this object was also used in the Simons proof above.

4 Variable Boundaries

The past examples of Plateau's problem has dealt with strict boundaries and their surface areas. We can introduce another factor into this equation by looking at variable boundaries. Specifically, imagine our boundaries to be like fishing line: boundaries can't be stretched or shrunk but are still flexible. Variable boundary Plateau problems are specifically called Euler-Plateau problems and have significant applications in physics like capillary flow and lightning construction.

One crucial idea in solving Euler-Plateau problems is the Euler-Plateau energy.

Definition 4.1 (Euler-Plateau Energy) *The energy is defined by the following equations for a given surface M.*

$$F[M] = \sigma \int_M dA + \oint_{\partial M} ds(\alpha\kappa^2 + \beta)$$

where

- *M is our surface*
- *σ is our surface tension*
- *2α is our surface rigidity*
- *β is our lagrange multiplier for line tension (enforces our boundary inextensibility $L = \oint_{\partial M} ds$)*

One way to intuitively understand energy is that it acts as a measure of how curvy a surface is (we see that Gaussian curvature is in the formula). If a surface is more curvy, we can expect it to have a larger and more complex minimizing surface. By minimizing our energy, we can find surfaces that adhere to our flexible boundary while still minimizing the surface area of our minimal surface. The physical interpretation of this can be directly linked to the physical shape of water droplets, where physical forces like surface tension and gravity both act together to give the water its shape.

5 Approachable Open Problems

During my research, I found a set of open ended problems that still have not been proven related to Plateau's problem. One that caught my eye was the following since it seemed to be understandable from my current background in analysis and geometry.

Conjecture 5.1 *Convex Curve Conjecture, Meeks Two convex jordan curves in parallel planes cannot bound a compact minimal surface of positive genus.*

This problem seems intuitive at first but is quite difficult to prove, which reminds me of the Jordan Curve Theorem when I first approached it. So far, we know that this conjecture holds for when our Jordan curves are extremal (i.e. on the boundary of their convex hull).