Symplectic Geometry and its Role in Classical Mechanics

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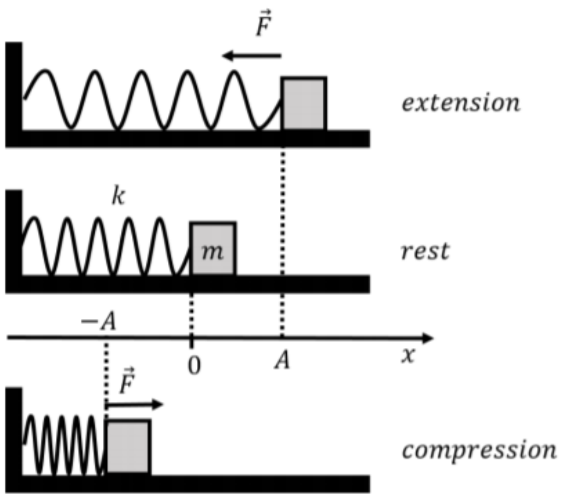
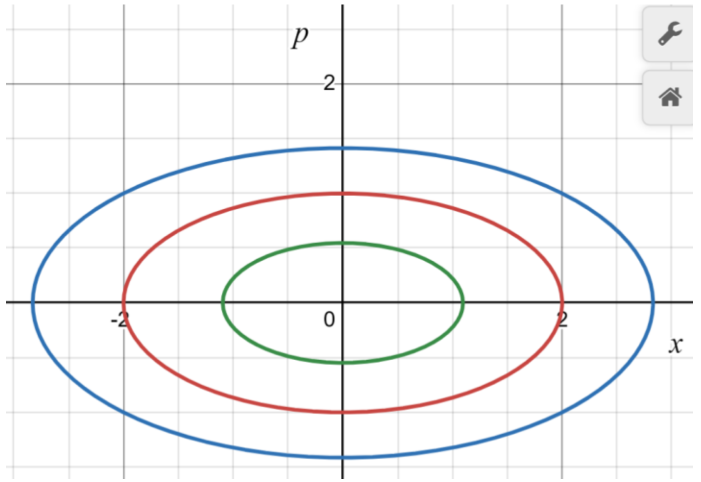
Phase Space



- In classical mechanics, we often study space of all possible positions

$$\sum F = ma$$

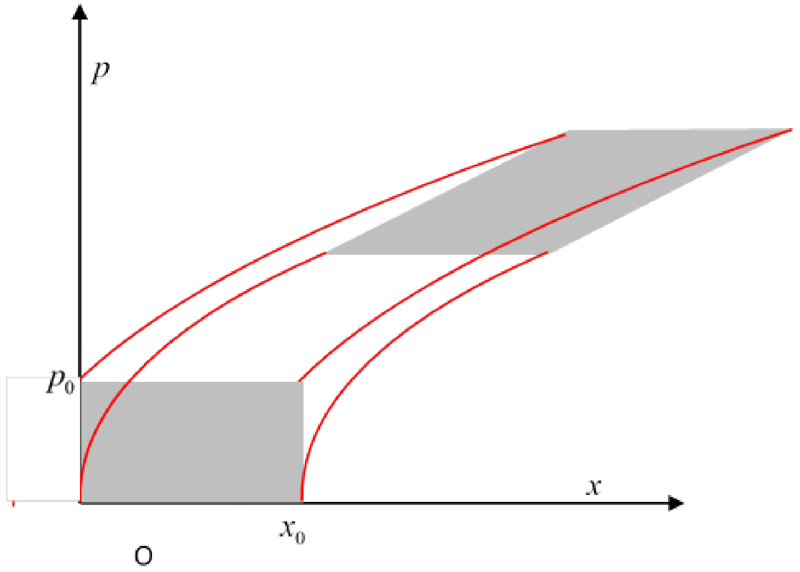
- In other formulations of mechanics, namely Hamiltonian mechanics, we instead study the space of all positions and momenta together, called ***phase space***



$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

Liouville's Theorem

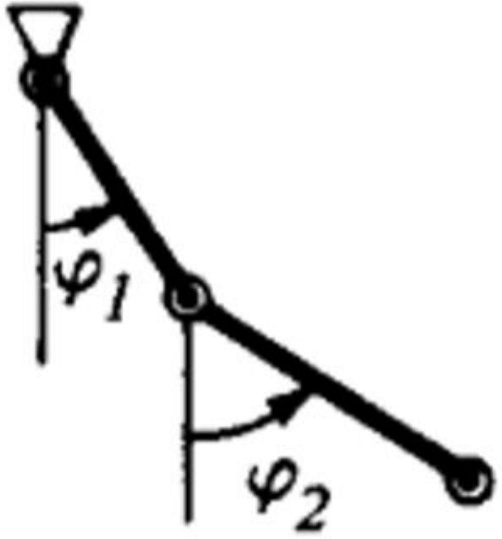
- Consider a region in phase space, i.e. some bounded set of points (x,p), at some initial time
- It will have some volume
- Then Liouville's theorem says that if you let each point evolve in time (so the region will move and warp), then the total volume will remain the exact same
- This is a more geometric theorem, thus to generalize this result, we will dive into more geometric generalizations of our notions of phase space

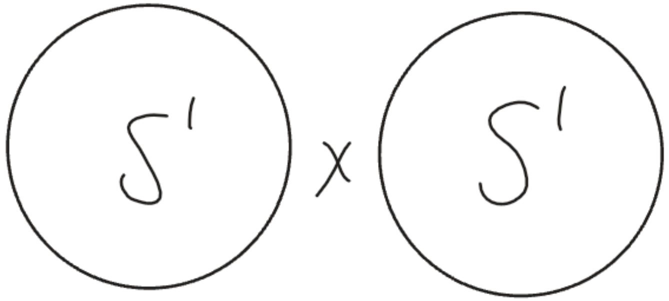


Four phase space trajectories for falling objects

Configuration Space

- The idea of a configuration space (the set of all possible positions of a system) can be defined as a differentiable manifold, which is something that locally looks like $\mathbb{R}^n$, but is overall warped/curved
- Instead of a 3-D cartesian coordinate, the configuration space can be defined in terms of "generalized coordinates"
- The dimension is the number of degrees of freedom





Tangent Space/Bundle

- You have…
  - a tangent line to a curve
  - a tangent plane to a surface
  - tangent space for a general differentiable manifold
- Given an n-dimensional configuration space M and point $\mathrm{x} \in M$, TM_x is the tangent space of M at x, i.e. the set of all tangent vectors based at x
  - ➔ TM_x is also n-dimensional
  - Intuition: $\mathrm{v} \in TM_x$ are the velocity vectors at position x
- The tangent bundle TM of a configuration space is the union of all tangent spaces and their base points
  - So if $\mathrm{x} \in M$ and $\mathrm{v} \in TM_x$, then $(\mathrm{x}, \mathrm{v}) \in TM$
  - ➔ TM is 2n-dimensional

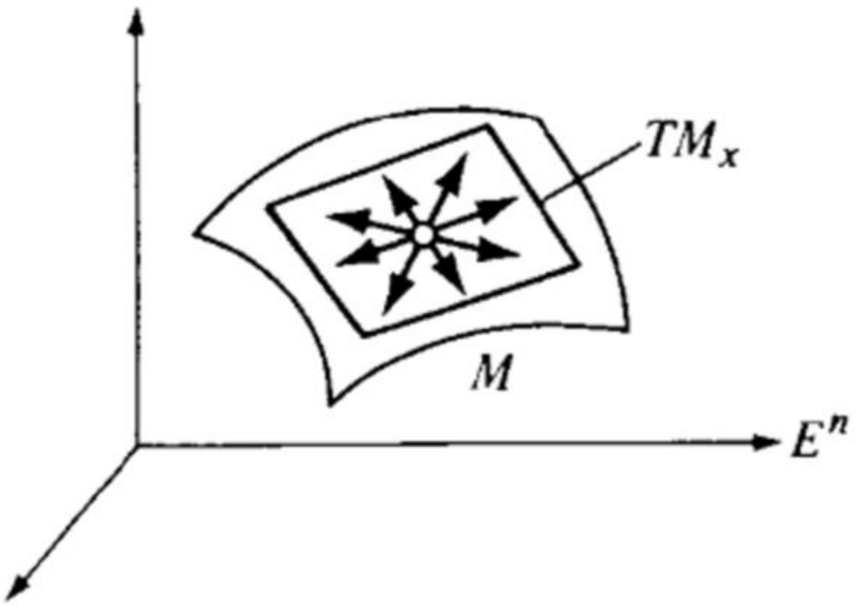


Figure 62 Tangent space

Cotangent Space/Bundle

- The tangent space at a point on a configuration space is a vector space
- Thus we may take its dual: the cotangent space
    - Given TM_x, the cotangent space is T^*M_x
- It's the space of 1-forms on the tangent space
  - Think of $\mathrm{v} \in TM_x$ as a column vector, and $\mathrm{p} \in T^*M_x$ as a row vector
  - Intuition: $\mathrm{p} \in T^*M_x$ are your momenta
- Similar to tangent bundles, the cotangent bundle T^*M is the union of all cotangent spaces and their base points
  - So if $\mathrm{x} \in M$ and $\mathrm{p} \in T^*M_x$, then $(\mathrm{x}, \mathrm{p}) \in T^*M$
  - ➔T^*M is 2n-dimensional
- So the cotangent bundle is just a generalization of phase space!

Symplectic 2-Form and Symplectic Manifolds

- Given a 2n-dimensional differentiable manifold M, a symplectic structure ω^2 is a closed nondegenerate 2-form on M
  - (M, ω^2) is called a symplectic manifold
- It turns out that the cotangent bundle (generalize phase space) has a natural symplectic structure, hence making it a symplectic manifold
- For a cotangent bundle T^*M with coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$, the symplectic structure is $\omega^2 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$
  - This is like the sum of projected areas in each 1d phase space
  - In fact by Darboux's theorem, $\forall (M, \omega^2)$, locally $\exists (q_1, \dots, q_n, p_1, \dots, p_n)\ s.t.$ $\omega^2 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$
  - In matrix form, $\omega^2 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

Hamiltonian Phase Flow

- Let M^{2n} be a 2n-dimensional symplectic manifold and suppose a particle starts with "position" $(\vec{q}_0, \vec{p}_0) \in M^{2n}$
  - This could be initial position and momentum on a cotangent bundle (phase space)
- Then the solution to Hamilton's eq'ns $(\vec{q}(t), \vec{p}(t))$ can be written as $\left(\vec{q}(t), \vec{p}(t)\right) = g^t(\vec{q}_0, \vec{p}_0)$, where g^t is 1-parameter group of diffeomorphisms called the ***Hamiltonian Phase Flow***
  - You can think of g^t as time-evolving whatever initial state it's applied to

A Generalization of Liouville’s Theorem

- Theorem: for any 2-chain c in M^{2n} (just think of a 2d surface in our cotangent bundle, i.e. generalized phase space), we have that ∀ time t

$$\int_c \omega^2 = \int_{g^t c} \omega^2$$

  - Intuition: the sum of the projected areas in each 1D phase space stays constant
- Let $(\omega^2)^k \coloneqq \omega^2 \wedge \omega^2 \wedge \cdots \wedge \omega^2$ k times
  - ➔ $(\omega^2)^n =$ volume form on M^{2n}
- Corollary: ∀k=1,…,n, and for any 2k-chain in M^{2n}, we have that ∀ time t

$$\int_c (\omega^2)^k = \int_{g^t c} (\omega^2)^k$$

  - ➔ In particular, volume of a region is constant in time, i.e. Liouville’s Theorem

Thank you!

- Thank you Tianyue for the very helpful insights and your flexibility
- And thank you to everyone who helped organize the DRP

Citations

- Pictures:
  - Slide 2 picture:
    - 13.1: The motion of a spring-mass system - Physics LibreTexts
  - Slide 3 picture:
    - Liouville (virginia.edu)
  - Slide 4 and 5 pictures:
    - A'rnold's Mathematical Methods of Classical Mechanics
- Readings:
  - A'rnold's Mathematical Methods of Classical Mechanics
  - Maxim Jeffs' Classical Mechanics