Scheming about Schemes

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May 2nd, 2024

Motivating Problem

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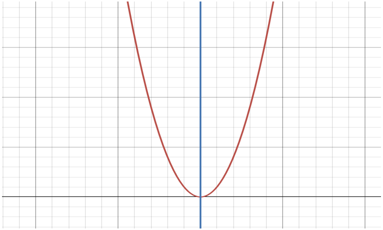

Figure: Graph of $y = x^2$ and $x^2 = 0$

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- So f and g intersect at a single point!

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- It's time to hatch a scheme!

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- We can now define a topology on $Spec(R)$ where $V(I)$ are our closed sets. This is the **Zariski Topology** on $Spec(R)$.

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- Since $k[x]$ is a PID, any ideal is of the form $I = (f(x))$ for some fixed polynomial f. $V(I)$ is just the set of linear factors of f, which is finite.
- Therefore, any closed set in $Spec(R)$ is a finite set of points or all of $Spec(R)$.

Zariski Topology (Example)

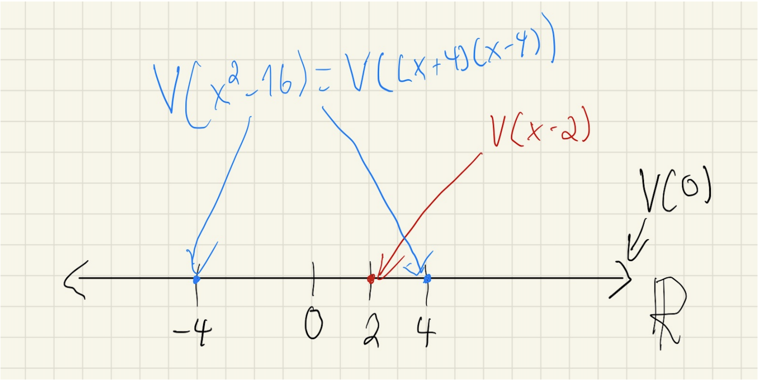

Figure: The Zariski Topology of $\mathbb{R}[x]$

Further Abstraction

- For many geometric spaces, we want to consider the functions defined locally on the space.
- E.g. Smooth Manifolds- for each open subset U, we consider the differentiable functions defined there.

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- A **pre-sheaf** on a topological space X is an assignment for each open set U, a set $\mathcal{F}(U)$ of "functions," together with restriction maps. For any open subset $V \subset U$, a map $\mathcal{F}(U) \to \mathcal{F}(V)$, compatible with composition.

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- A **sheaf** satisfies further properties consistent with our "functional" intuition.

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- For $U = X$, the algebraic functions that make sense on U are polynomials, i.e. elements $h \in k[x]$. Thus, $\mathcal{O}_X(X) = k[x]$.
- Let $g(x) := x - \alpha$ and $U = X \setminus V(g)$, i.e. all points besides $x - \alpha$. Functions that are well defined here are of the form

$$\frac{f(x)}{(x-\alpha)^n}$$

for $f \in k[x]$ and n a natural number. Formally, we have $\mathcal{O}_X(U) = k[x]_g$.

Scheme Definition

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- A **scheme** is a topological space which has an open covering of affine schemes.

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- R' is isomorphic to $k[x]/x^2$. To count the multiplicity, we find the dimension of R' as a k vector space, which is 2.
- Therefore the scheme theoretic language captures this multiplicity, and much more!