A Logic for Language: Exploring the Lambek Calculus

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  = computer programs
- **The Miracle (Curry–Howard):**
  mathematical proofs = computer programs :)

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- $\dfrac{P \qquad P \rightarrow Q}{Q}$

Proofs are trees

- "P. If P, then Q. If Q, then R. Therefore, R."

$$P, P \to Q, Q \to R \vdash R$$

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- Reference picture, in case it's been a while since you left DRL:

Figure 1: Happy Earth week!

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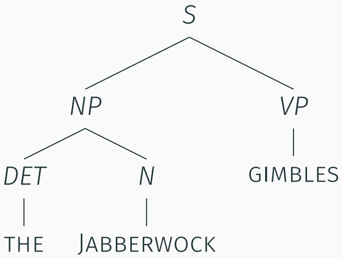

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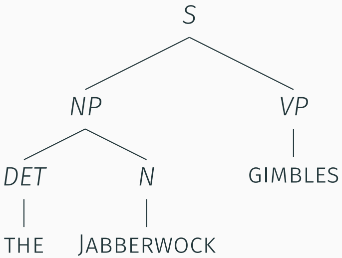
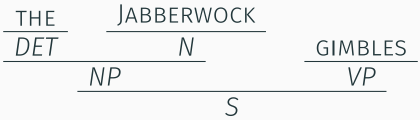


- As a right-side-up tree:



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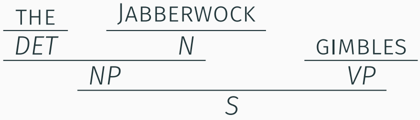
- When a linguistic tree…



What if…

- When a linguistic tree…

$$\dfrac{\dfrac{\dfrac{\text{THE}}{DET}\qquad\dfrac{\text{JABBERWOCK}}{N}}{NP}\qquad\dfrac{\text{GIMBLES}}{VP}}{S}$$

- …becomes a proof!

$$\dfrac{\dfrac{\dfrac{\text{THE}}{N \rightarrow NP}\qquad\dfrac{\text{JABBERWOCK}}{N}}{NP}\qquad\dfrac{\text{GIMBLES}}{NP \rightarrow S}}{S}$$

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- So saying that a sentence is grammatical is the same as saying that if we have the parts of the sentence as assumptions, we can prove S

$$x_1 \ldots x_n \text{ is grammatical} \iff x_1, \ldots, x_n \vdash S$$

Complications

- Some problems with orderings:

$$P_1, P_2, P_3 \vdash Q \implies P_2, P_3, P_1 \vdash Q,$$

but

$$\text{the}, \text{dog}, \text{runs} \vdash S \not\Longrightarrow \text{dog}, \text{runs}, \text{the} \vdash S.$$

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- Some problems with adding assumptions:

$$P_1, P_2, P_3 \vdash Q \implies P_1, P_2, P_3, P_4 \vdash Q,$$

but

$$\text{the}, \text{dog}, \text{runs} \vdash S \not\Longrightarrow \text{the}, \text{dog}, \text{runs}, \text{faucet} \vdash S.$$

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- Features one implication arrow for each direction!

$$\frac{A \qquad A \rightarrow B}{B} \qquad \frac{\begin{matrix}\emptyset\ [A] \\ \vdots \\ B\end{matrix}}{A \rightarrow B} \qquad \frac{B \leftarrow A \qquad A}{B} \qquad \frac{\begin{matrix}[A]\ \emptyset \\ \vdots \\ B\end{matrix}}{B \leftarrow A}$$

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- A substructural logic where order matters
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- We can use this to fix our proof from earlier:

$$\dfrac{\dfrac{\dfrac{\text{The}}{NP \leftarrow N} \qquad \dfrac{\text{Jabberwock}}{N}}{NP} \qquad \dfrac{\text{gimbles}}{NP \to S}}{S}$$

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Function application

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Function definition

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Function application	Function definition
$$\frac{A \qquad A \to B}{B}$$	$$\frac{\begin{matrix} \emptyset\ [A] \\ \vdots \\ B \end{matrix}}{A \to B}$$

- This is the Curry–Howard correspondence: we've turned a proof into the instructions of a program!

Deus ex machina

- Example from before:

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- Translation:

$$\mathrm{gimbles}(\mathrm{the}(\mathrm{Jabberwock}))$$

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- Translation:

$$\mathrm{gimbles}(\mathrm{the}(\mathrm{Jabberwock}))$$

- $\mathrm{Jabberwock}$ = the set of all Jabberwocks,
  $\mathrm{the}(S)$ = the sole member of S,
  $\mathrm{gimbles}(x)$ = true if x gimbles, false otherwise.
- Sentences are programs that compute whether to output “true” or “false” based on the circumstances

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- Two interpretations (skipping some proof steps):

$$\dfrac{\dfrac{\text{EVERY STUDENT}}{S \leftarrow (NP \rightarrow S)} \qquad \dfrac{\dfrac{\text{STUDIES}}{(NP \rightarrow S) \leftarrow NP} \qquad \dfrac{\text{A PROOF}}{(S \leftarrow NP) \rightarrow S}}{NP \rightarrow S}}{S}$$

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A tale of two trees (continued)

- $$\dfrac{\dfrac{\textsc{every student}}{S \leftarrow (NP \rightarrow S)} \qquad \dfrac{\dfrac{\textsc{studies}}{(NP \rightarrow S) \leftarrow NP} \qquad \dfrac{\textsc{a proof}}{(S \leftarrow NP) \rightarrow S}}{NP \rightarrow S}}{S}$$

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A tale of two trees (continued)

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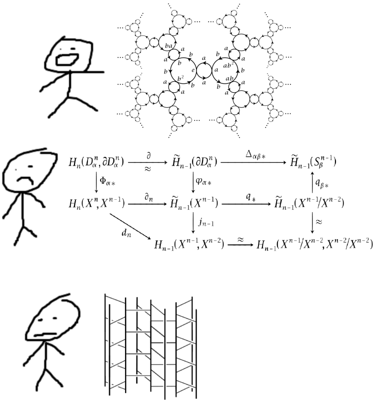
- Curry–Howard says:

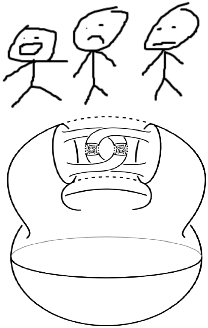
$$\begin{cases} \text{true} & \forall s : \text{student},\ \exists p : \text{proof},\ s \text{ studies } p \\ \text{false} & \text{otherwise} \end{cases}$$

vs

$$\begin{cases} \text{true} & \exists p : \text{proof},\ \forall s : \text{student},\ s \text{ studies } p \\ \text{false} & \text{otherwise} \end{cases}$$



(a) $\forall s,\ \exists p,\ s$ studies p



(b) $\exists p,\ \forall s,\ s$ studies p

Figure 2: MATH 6010 students reading Allen Hatcher's *Algebraic Topology*

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- Syntax = mathematical proofs
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- Mathematical proofs = computer programs (Curry–Howard)
- Mathematical logic is amazing!