Notes for Graduate Student Seminar: Lang's Thesis

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Abstract

The Brauer group $\mathrm{Br}(k)$ of a field k is an important object of study in number theory. A convenient condition to force the Brauer group to be trivial is for k to be a quasi-algebraically closed field. Quasi-algebraically closed fields are part of a more general notion of C_i fields, introduced in Lang's thesis in 1951. The goal of this talk is to prove that finite fields $\mathbb{F}_q$ and function fields $\bar{k}(t)$ over algebraically closed fields are quasi-algebraically closed hence have trivial Brauer groups.

Disclaimer. These notes follow [2], [3], and [4] verbatim.

Motivation

Representation of groups as matrices. To understand a finite group G, we can act it on a finite-dimensional vector space V, so that G is now represented by matrices[1] in $\mathrm{GL}(V)$. Two representations $G \to \mathrm{GL}(V)$ and $G \to \mathrm{GL}(W)$ give rise to a third representation $G \to \mathrm{GL}(V \oplus W)$ by:

$$\begin{matrix} g \leftrightarrow A \in \mathrm{GL}(V) \\ g \leftrightarrow B \in \mathrm{GL}(W) \end{matrix} \implies g \leftrightarrow \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathrm{GL}(V \oplus W).$$

Conversely, given a representation $G \to \mathrm{GL}(V)$, we want to decompose it into irreducible blocks[2]. This is provided by:

[1] Upon a choice of basis

[2] These are called the irreducible representations.

Theorem 1 (Maschke, Artin-Wedderburn)**.** Let G be a finite group. Let k be a field whose characteristic does not divide $|G|$. Then:

1. The group ring $k[G]$ is isomorphic to the product of matrix rings $M_{n_i}(D_i)$ where D_i are division algebras that are finite-dimensional over k.

2. The action of G on these k-vector spaces $M_{n_i}(D_i)$ are precisely all the irreducible representations of G on finite-dimensional k-vector spaces.

Example. The finite-dimensional division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. Consider $\mathbb{R}[\mathbb{Z}/3\mathbb{Z}]$. This is a 3-dimensional $\mathbb{R}$-vector space, so either:

- $\mathbb{R}[\mathbb{Z}/3\mathbb{Z}] \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ or
- $\mathbb{R}[\mathbb{Z}/3\mathbb{Z}] \cong \mathbb{R} \times \mathbb{C}$.

But $\mathbb{R}[\mathbb{Z}/3\mathbb{Z}]$ has only one $\mathbb{R}[\mathbb{Z}/3\mathbb{Z}]$-invariant subspace of dimension 1 so $\mathbb{R}[\mathbb{Z}/3\mathbb{Z}] \cong \mathbb{R} \times \mathbb{C}$. Thus $\mathbb{R}[\mathbb{Z}/3\mathbb{Z}]$ has two irreducible representations over $\mathbb{R}$.

Therefore, to classify the irreducible representations over k, one first classifies the finite-dimensional division algebras over k. We store this information as follows.

Define the **Brauer group of a field** k to be the set of (isomorphism classes of) division algebras which are finite-dimensional over k and have center k. The Brauer group of k is denoted $\mathrm{Br}(k)$.

For example, $\mathrm{Br}(\mathbb{C}) = \{\mathbb{C}\}$ and $\mathrm{Br}(\mathbb{R}) = \{\mathbb{R}, \mathbb{H}\}$.

We define the group structure on $\mathrm{Br}(k)$ in the following manner. Given two division algebras $D_1, D_2 \in \mathrm{Br}(k)$, it turns out that $D_1 \otimes_k D_2 \cong M_d(D_3)$ for some $D_3 \in \mathrm{Br}(k)$. Thus define the group multiplication in $\mathrm{Br}(k)$ by $[D_1] \cdot [D_2] = [D_3]$.

The Brauer group under field extension. Let k'/k be a field extension. Given a division algebra $D \in \mathrm{Br}(k)$, the tensor product $D \otimes_k k'$ is isomorphic to some matrix ring $M_d(D')$ with $D' \in \mathrm{Br}(k')$. There are two consequences to this.

First we have $D \otimes_k \bar{k} \cong M_d(\bar{k})$. This can be improved:[3] there exists a Galois field extension k'/k for which $D \otimes_k k' \cong M_d(k')$.[4]

[3] See Corollary 2.2.6 in [1].

[4] Such a field k' is said to **split** the division algebra D.

Second, $\mathrm{Br}(-)$ can be viewed as a functor **Fields** $\to$ **Groups** taking a morphism of fields $k \hookrightarrow k'$ to

$$\begin{aligned} \mathrm{Br}(k) &\to \mathrm{Br}(k') \\ [D] &\mapsto [D'], \end{aligned}$$

where D' is the division algebra for which $D \otimes_k k' \cong M_d(D')$. Thus $\mathrm{Br}(-)$ is a functorial way to tell fields apart.

Finite-dimensional division algebra over its center has square dimension. Let $D \in \mathrm{Br}(k)$ be a division algebra. We claim that $\dim_k D$ is a square number. Under the natural injection $D \hookrightarrow D \otimes_k \bar{k} \cong M_d(\bar{k})$, a k-basis $\{\omega_1, \ldots, \omega_r\}$ for D remains a $\bar{k}$-basis for $D \otimes_k \bar{k}$. Therefore:

$$\dim_k D = \dim_{\bar{k}}(D \otimes_k \bar{k}) = \dim_{\bar{k}} M_d(\bar{k}) = d^2.$$

C_i Fields

Another way to tell fields apart is by a property. Define a field k to be:

- C_0 if any degree d form[5] over k in $n > 1$ variables has a nontrivial zero.
- C_1 if any degree d form over k in $n > d^1$ variables has a nontrivial zero.
- C_i if any degree d form over k in $n > d^i$ variables has a nontrivial zero.

[5] By a **form over** k, we just mean a **homogeneous polynomial over** k.

One can show that a field is C_0 if and only if it is an algebraically closed field.[6] Properties about C_1 fields were discussed and known by Artin, Tsen, and others under the label *quasi-algebraically closed fields*. Lang generalized this to the notion of C_i fields.

[6] One direction is clear. Conversely, we show that $k \neq \bar{k}$ implies that k is not C_0. This will follow from Lemma 4 below.

Proposition 2. If a field k is C_1, then $\mathrm{Br}(k)$ is trivial.

Proof. We prove the contrapositive. Let $D \in \mathrm{Br}(k)$ be a division algebra of dimension $d^2 > 1$ over k. Let $\{\omega_1, \ldots, \omega_{d^2}\}$ be a k-basis for D. Let k'/k be a Galois field extension which splits D i.e. $D \otimes_k k' \cong M_d(k')$. Then the injective k-algebra homomorphism

$$\varphi : D \hookrightarrow D \otimes_k k' \cong M_d(k')$$

sends a generic element $x = x_1\omega_1 + \cdots + x_{d^2}\omega_{d^2} \in D$ to a $d \times d$ matrix $[x]$ over k'. Consider the polynomial

$$g(x_1, \ldots, x_{d^2}) = \det([x])$$

in the variables $x_1, \ldots, x_{d^2}$.

Let $A_i = [\omega_i]$ so $g(x_1, \ldots, x_{d^2}) = \det(x_1A_1 + \cdots + x_{d^2}A_{d^2})$.

<u>Claim 1.</u> g is a homogeneous polynomial of degree d.

<u>Proof.</u> Clearly $g(\lambda x_1, \ldots, \lambda x_{d^2}) = \lambda^d g(x_1, \ldots, x_{d^2})$.

<u>Claim 2.</u> g depends on all the variables $x_1, \ldots, x_{d^2}$.

<u>Proof.</u> We have $g(0, 0, \ldots, 0) = 0$ and $g(1, 0, \ldots, 0) = \det A_1 \neq 0$ since $\omega_1 \in D$ is invertible and φ is a k-algebra homomorphism. So g depends on the variable x_1. Similarly for x_i.

<u>Claim 3.</u> $g \in k[x_1, \ldots, x_{d^2}]$.

<u>Proof.</u> Let $\sigma \in \mathrm{Gal}(k'/k)$. By Skolem-Noether,

$$\sigma \cdot g(x) = \det(x_1\sigma(A_1) + \cdots + \sigma(A_{d^2})) = \det(x_1BA_1B^{-1} + \cdots + x_{d^2}BA_{d^2}B^{-1}) = g(x)$$

for some $B \in GL_d(k')$.

If $x \neq 0$, $\det([x]) \neq 0$ so g only has the trivial zero. Therefore k is not C_1. ∎

C_i *fields: examples*

We now see that certain fields are C_1, hence have trivial Brauer groups. This section follows [4], section 3 of [3], and chapter 3 of [2] verbatim.

Proposition 3. Finite field $\mathbb{F}$ is C_1.

Chevalley-Warning theorem; proof via Wikipedia

Proof. Let $|\mathbb{F}| = q = p^k$. Let $f(x_1, ..., x_n)$ be a degree d form over $\mathbb{F}$ with $n > d$. We show that f has a nontrivial zero $\underline{x} \in \mathbb{F}^n$.

If $i < q - 1$ then

$$\sum_{x \in \mathbb{F}} x^i = 0$$

so the sum over $\mathbb{F}^n$ of any polynomial in $x_1, ..., x_n$ of degree less than $n(q-1)$ also vanishes.

We have the indicator function

$$1 - f(\underline{x})^{q-1} = \begin{cases} 1 \text{ if } x \text{ is a zero of } f, \\ 0 \text{ else.} \end{cases}$$

Thus

$$\# \text{ zeros of } f = \sum_{\underline{x} \in \mathbb{F}^n} (1 - f(\underline{x})^{q-1}) \equiv 0 \pmod p.$$

As $0 \in \mathbb{F}^n$ is a zero of f, by above, f has a nontrivial zero. ■

Our goal is now to show $\bar{k}(t)$ is C_1. This requires introducing the concept of a normic form and a key theorem due to Lang and Nagata.

Let K/k be a finite field extension of degree $n > 1$ and consider a k-basis $\{\omega_1, ..., \omega_n\}$ for K. A generic element $x \in K$ can be written as

$$x = x_1\omega_1 + \cdots + x_n\omega_n.$$

Consider the left multiplication map $l_x : K \to K : y \mapsto xy$. As a k-linear map, l_x has a matrix representation $[l_x]$ with respect to our chosen basis. Its determinant $\det[l_x]$ is then a homogeneous polynomial of degree n in the n variables $x_1, ..., x_n$. This polynomial is called the **norm form of** K/k.

Example. Consider the field extension $\mathbb{C}/\mathbb{R}$. Viewing $\mathbb{C}$ as an $\mathbb{R}$-vector space, the map given by multiplication by $x + iy$ has the matrix representation

$$[l_{x+iy}] = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

in the $\mathbb{R}$-basis $\{1, i\}$. Therefore the norm form for $\mathbb{C}$ over $\mathbb{R}$ is

$$\det \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = x^2 + y^2.$$

More generally, a **normic form over a field** k is a homogeneous polynomial ϕ over k of degree d in n variables such that $n = d$ and ϕ has only the trivial zero over k.

Example. For the field extension $\mathbb{C}/\mathbb{R}$, we can use the norm form $N(x, y) = x^2 + y^2$ to generate more normic forms by plugging N into itself. For example,

$$N(N(x_1, x_2), N(y_1, y_2)) = (x_1^2 + x_2^2)^2 + (y_1^2 + y_2^2)^2$$

is a normic form. Therefore:

Lemma 4. Let k be a field that is not algebraically closed. Then there exist normic forms over k of arbitrarily large degree. ∎

Theorem 5 (Lang-Nagata)**.** Let k be a C_i field. Let $f_1, ..., f_r$ be homogeneous polynomials over k of degree d in n variables. If $n > rd^i$, then they have a nontrivial common zero in k^n.

Proof. The $i = 0$ case corresponds to k being algebraically closed. This is handled by classical algebraic geometry; see Proposition 7.2 of Chapter 1 in Hartshorne.

Therefore, let us now assume that k is not algebraically closed.

Let $i = 1$. Let ϕ be a normic form of degree $e \geq r$. Consider the form

$$\phi^{(1)} = \phi(f_1, ..., f_r | f_1, ..., f_r | \cdots | f_1, ..., f_r | 0, ..., 0)$$

where after each slash we use new variables, and we insert as many complete sets of f's as possible.

Thus $\phi^{(1)}$ has $n \left\lfloor \frac{e}{r} \right\rfloor$ variables and has degree $de \leq dr \left(\left\lfloor \frac{e}{r} \right\rfloor + 1 \right)$. If k is C_1, we want

$$n \left\lfloor \frac{e}{r} \right\rfloor > dr \left(\left\lfloor \frac{e}{r} \right\rfloor + 1 \right)$$

or

$$(n - dr) \left\lfloor \frac{e}{r} \right\rfloor > dr.$$

Since k is C_1, $n - dr > 0$, so the above inequality holds for large enough e. Then $\phi^{(1)}$ has a nontrivial zero. Since ϕ is normic, it is a common zero of all the f's.

For $i > 1$, see Theorem 3.3.7 of [3] or Theorem 3.4 of [2]. ∎

Two consequences of this theorem are:

Theorem 6. If k is a C_i field, then every algebraic extension of k is also a C_i field. Consequence 1 of Lang-Nagata

Proof. (Lang) It suffices to prove the theorem for a finite extension K of k, since the coefficients of any given form lies in a finite extension.

Let K be a finite field extension of k of degree e. Let $\{\omega_1, ..., \omega_e\}$ be a k-basis of K. Let $f(\underline{x})$ be a form over K of degree d and in $n > d^i$ variables. Introduce new variables $\overline{x_{ij}}$ with respect to our basis, such that

$$x_i = \overline{x_{i1}}\omega_1 + \cdots + \overline{x_{ie}}\omega_e$$

for $i = 1, ..., n$. Then

$$f(\underline{x}) = f_1(\overline{x})\omega_1 + \cdots + f_e(\overline{x})\omega_e$$

for some forms f_i of degree d in n variables. Since $en > ed^i$, by the previous theorem, these forms f_i have a nontrivial common zero $\bar{x}$ over K. This gives a nontrivial zero x for f over K. ■

Next we have

Theorem 7. If k is a C_i field, and K is an extension of k of finite transcendence degree j, then K is a C_{i+j} field.

Consequence 2 of Lang-Nagata

Proof. (Tsen, essentially) By the previous theorem, we are reduced to purely transcendental extensions. By induction on j, we are reduced to the case $K = k(t)$. By homogeneity, it suffices to consider forms with coefficients in the polynomial ring $k[t]$.

Let f be a form of degree d in $n > d^{i+1}$ variables with coefficients in $k[t]$. Introduce new variables $\overline{x_{ij}}$ with

$$x_i = \overline{x_{i0}} + \overline{x_{i1}}t + \overline{x_{i2}}t^2 + \cdots + \overline{x_{is}}t^s$$

where s will be chosen later. If r is the highest degree of the coefficients of f, we get

$$f(x) = f_0(\overline{x}) + f_1(\overline{x})t + \cdots + f_{ds+r}(\overline{x})t^{ds+r}$$

where each form f_i is of degree d in $n(s+1)$ variables.

We finish by applying Theorem 5 to these forms f_i. For large enough s, the inequality

$$n(s+1) > d^i(ds + r + 1)$$

holds, since this is the same as

$$(n - d^{i+1})s > d^i(r+1) - n.$$

Therefore the f_i have a nontrivial common zero. This gives a nontrivial zero to f over $k(T)$. ■

As a corollary, since $\bar{k}$ is a C_0 field, we deduce that $\bar{k}(t)$ is a C_1 field.

This is the original Tsen's theorem.

Final remarks

Lemma 8. Let k be a field complete under a non-archimedean valuation and let its ring of integers $\mathfrak{o}$ be compact. Let f be a form over F and suppose that there exists a sequence of forms f_i converging to f such that each f_i has a nontrivial zero in k. Then f also has a nontrivial zero in k.

See Lemma on page 379 of [4].

Proof. We may assume all coefficients and zeros are integers i.e. elements of $\mathfrak{o}$. By homogeneity, we may assume that some α_i in a zero $(\alpha_1, \ldots, \alpha_n)$ of f_i is a unit. We obtain in this way a collection of n-tuples in the Cartesian product $\mathfrak{o} \times \cdots \times \mathfrak{o}$. This product is compact, so our collection has an accumulation point which will be our desired zero. It is nontrivial since each element of our collection has at least one unit component. ■

Theorem 9. Let $\mathbb{F}$ be a finite field. Then $\mathbb{F}((t))$ is a C_2 field.

See Theorem 8 of [4].

Proof. Given the form $f(x_1, \ldots, x_n)$ of degree d, $n > d^2$ with integral coefficients, we must show it has a nontrivial zero. If we omit the coefficients of f after a power of t, we obtain a form over $\mathbb{F}[t]$. As the field $\mathbb{F}(t)$ is C_2, this form has a nontrivial zero. Doing this for each power of t we obtain an approximating sequence of forms, each having a zero. We can apply the above lemma to complete the proof. ■

Following the analogy between function fields and number fields, Artin conjectured that $\mathbb{Q}_p$ is C_2. This turned out to be false: Guy Terjanian constructed an explicit counterexample over $\mathbb{Q}_2$ in 1966. However, Ax and Kochen managed to prove the following surprising statement: for any integer $d > 0$, there exists a finite set of primes $P(d)$ such that the C_2 property holds for forms of degree d in $\mathbb{Q}_p$ for all $p \notin P(d)$.

The proof of the Ax-Kochen theorem uses model theory. For more on this, see [3].

References

[1] Philippe Gille, Tamás Szamuely (2006). *Central simple algebras and Galois cohomology*. Cambridge Studies in Advanced Mathematics. **101**. Cambridge: Cambridge University Press.

[2] Marvin J. Greenberg. *Lectures of forms in many variables*. Mathematics Lecture Note Series. New York-Amsterdam: W.A. Benjamin.

[3] Alex Kruckman, "The Ax-Kochen Theorem: An Application of Model Theory to Algebra" (Undergraduate honors thesis).

[4] Serge Lang. "On Quasi-algebraic closure", *Annals of Mathematics*, **55**: 373-390.