Exam to determine the Penn Putnam team for 2016.
Do as many problems as you can.
Time: 90 minutes.

Name: Solutions

E-mail: ______________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

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1. (10 points) Let x, y, z be positive real numbers, such that $x + y + z = 3$. What is the smallest possible value of $f(x, y, z)$, where

$$f(x, yz) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \qquad ?$$

AM $\leftrightarrow$ GM inequality:

$3 = x + y + z \geq 3\sqrt[3]{xyz} \Rightarrow xyz \leq 1$

$1/x + 1/y + 1/z \geq 3\sqrt[3]{\frac{1}{xyz}} \geq 3$

value achieved at $x = y = z = 1$, so

$\boxed{\min f = 3}$.

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2. (10 points) Let f be a polynomial with integer coefficients. Suppose that there exists $k \in \mathbb{N}, k \geq 2$, such that none of the numbers $f(1), \ldots, f(k)$ is divisible by k.

   Prove that f has no integer roots, i.e. there is no integer m, such that $f(m) = 0$.

① If $f(x) = a_0 + a_1 x + \ldots + a_d x^d$,

since $a^i \equiv b^i \pmod{k}$ if $a \equiv b \pmod{k}$

$\Rightarrow$ If $a \equiv b \Rightarrow f(a) \equiv f(b) \pmod{k}$

② Suppose $f(m) = 0 \Rightarrow$ ~~[illegible]~~

Let $m \equiv k.p + q$, $q \in \{1, \ldots, k\}$ $\rightarrow$ residue of m mod k

$\Rightarrow 0 = f(m) \equiv f(kp+q) \equiv f(q) \pmod{k}$.

But $k \nmid f(q)$ $\Rightarrow$ contradiction, $f(m) \neq 0$ $\forall m \in \mathbb{Z}$.

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3. (10 points) Let n be an integer and $S \subset \{1, 2, \ldots, 2n\}$ with $n+1$ elements. Prove that there exist two distinct elements $a, b \in S$, such that $a|b$.

For each odd integer $2m+1$, let

$$R_{2m+1} = \{ 1(2m+1), 2.(2m+1), 2^2(2m+1), \ldots \} \subset \{1, \ldots, 2n\}$$

$$= \{ 2^i.(2m+1) \mid i = 0, \ldots, 2^i(2m+1) \leq 2n \}.$$

Then $R_{2a+1} \cap R_{2b+1} = \phi$ if $a \neq b$,

$\underline{\text{else}}$ $2^i(2a+1) = 2^j(2b+1)$

if $i \neq j$, $\cancel{\Rightarrow 2}$ say $i > j \rightarrow$ $2b+1 = 2^{i-j}(2a+1) \rightarrow$ even

odd

$\Rightarrow i = j \rightarrow a = b$

So $S = [R_1 \cap S] \sqcup [R_3 \cap S] \sqcup \ldots \sqcup [R_{2n-1} \cap S]$.

n disjoint sets

$|S| \geq n+1 \Rightarrow \exists m$, s.t. $|R_{2m+1} \cap S| \geq 2$

$$\{ \underset{a}{2^i(2m+1)}, \underset{b}{2^j(2m+1)} \ldots \}$$

✓

4. (10 points) Express the following sum as one rational function (i.e. ratio of two polynomials):

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1-x^{2^{n+1}}}$$

1st solution: telescoping:

$$x^{2^n} = 1 + x^{2^n} - 1$$

$$F_k = \sum_{n=0}^{k} \frac{x^{2^n}}{1-x^{2^{n+1}}} = \sum_{n=0}^{k} \frac{1+x^{2^n}-1}{1-x^{2^{n+1}}} =$$

$$= \sum_{n=0}^{k} \frac{1+x^{2^n}}{\underbrace{1-x^{2^{n+1}}}_{(1-x^{2^n})(1+x^{2^n})}} - \sum_{n=0}^{k} \frac{1}{1-x^{2^{n+1}}} =$$

$$\underline{n'=n+1}$$

$$= \sum_{n=0}^{k} \frac{1}{1-x^{2^n}} - \sum_{n'=1}^{k+1} \frac{1}{1-x^{2^{n'}}} =$$

cancel.

$$= \frac{1}{1-x^{2^0}} - \frac{1}{1-x^{2^{k+1}}}.$$

$$\lim_{k\to\infty} F_k = \frac{1}{1-x} - \underbrace{\lim_{k\to\infty} \frac{1}{1-x^{2^{k+1}}}}_{1} = \frac{1}{1-x} - 1 = \frac{x}{1-x}.$$

2nd solution:

$$\frac{x^{2^n}}{1-x^{2^{n+1}}} = \sum_{m=0}^{\infty} x^{2^n} \cdot x^{2^{n+1}\cdot m} = \sum_{m=0}^{\infty} x^{2^n(1+2m)}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{x^{2^n}}{1-x^{2^{n+1}}} = \sum_{n,m} x^{2^n(1+2m)} \leftarrow \text{each integer appears once.} = x + x^2 + \ldots = \frac{x}{1-x}$$

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5. (10 points) Let a be a positive real number. Find

$$\int_{x=0}^{\infty} \frac{\arctan(ax) - \arctan(x)}{x}\,dx.$$

Let $f(a) = \int_{x=0}^{\infty} \frac{\arctan(ax) - \arctan(x)}{x}\,dx$

$\Rightarrow f'(a) = \int_{x=0}^{\infty} \frac{x}{x(1+a^2x^2)}\,dx = \int_{x=0}^{\infty} \frac{1}{1+a^2x^2}\,dx$

$= \frac{1}{a} \int_{ax=0}^{\infty} \frac{d\,ax}{1+(ax)^2} = \frac{1}{a} \arctan(ax)\Big|_0^{\infty} = \frac{1}{a}(\pi/2 - 0)$

$\Rightarrow f'(a) = \frac{\pi}{2}\frac{1}{a}$

$\Rightarrow f(a) = \int_1^a \frac{\pi}{2}\frac{1}{y}\,dy + \underset{\int 0\,dx = 0}{\underset{\shortparallel}{f(1)}} = \boxed{(\ln a)\pi/2}$.