Solving DSGE Models

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Solving DSGE Models

- A variety of numerical techniques are available to solve rational expectations systems.

- In the context of likelihood-based DSGE model estimation linear approximation methods are very popular because they lead to a state-space representation of the DSGE model that can be analyzed with the Kalman filter.

- Log-linearization of $f(x)$:

  1. write $f(x) = f(e^z)$;

  2. conduct a first-order Taylor approximation around x_0 in terms of z:

$$f(e^{\ln x}) \approx f(x_0) + x_0 f^{(1)}(x_0)(\ln x - \ln x_0).$$

Solving DSGE Models

- Our DSGE model leads to:

$$\begin{aligned}
\hat{y}_t &= I\!E_t[\hat{y}_{t+1}] + \hat{g}_t - I\!E_t[\hat{g}_{t+1}] - \frac{1}{\tau}\left(\hat{R}_t - I\!E_t[\hat{\pi}_{t+1}] - I\!E[\hat{z}_{t+1}]\right) && (1)\\
\hat{\pi}_t &= \beta I\!E_t[\hat{\pi}_{t+1}] + \kappa(\hat{y}_t - \hat{g}_t) && (2)\\
\hat{c}_t &= \hat{y}_t - \hat{g}_t, && (3)\\
\hat{R}_t &= \rho_R \hat{R}_{t-1} + (1-\rho_R)\psi_1 \hat{\pi}_t + (1-\rho_R)\psi_2\,(\hat{y}_t - \hat{g}_t) + \epsilon_{R,t} && (4)\\
\hat{g}_t &= \rho_g \hat{g}_{t-1} + \epsilon_{g,t} && (5)\\
\hat{z}_t &= \rho_z \hat{z}_{t-1} + \epsilon_{z,t} && (6)
\end{aligned}$$

where

$$\kappa = \tau\frac{1-\nu}{\nu\pi^2\phi}.$$

- Introduce $\eta_t^\pi = \hat{\pi}_t - I\!E_{t-1}[\hat{\pi}_t]$ and $\eta_t^y = \hat{y}_t - I\!E_{t-1}[\hat{y}_t]$.

Solving DSGE Models

- Linearized DSGE leads to linear rational expectations (LRE) system:

$$\Gamma_0(\theta)s_t = \Gamma_1(\theta)s_{t-1} + \Psi\epsilon_t + \Pi\eta_t \tag{7}$$

  where

  - s_t is a vector of model variables, ϵ_t is a vector of exogenous shocks,
  - η_t is a vector of RE errors with elements $\eta_t^x = \hat{x}_t - \mathbb{E}_{t-1}[\hat{x}_t]$, and
  - s_t contains (among others) the conditional expectation terms $\mathbb{E}_t[\widetilde{x}_{t+1}]$.

- Solution methods for LREs: Blanchard and Kahn (1980), King and Watson (1998), Uhlig (1999), Anderson (2000), Klein (2000), Christiano (2002), Sims (2002).

- Overall the solution in terms of s_t is of the form

$$s_t = \Phi_1(\theta)s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t. \tag{8}$$

Example

- Consider the simplified system:

$$\hat{y}_t = I\!\!E_t[\hat{y}_{t+1}] - \frac{1}{\tau}(\hat{R}_t - I\!\!E_t[\hat{\pi}_{t+1}]) \tag{9}$$

$$\hat{\pi}_t = \beta I\!\!E_t[\hat{\pi}_{t+1}] + \kappa \hat{y}_t \tag{10}$$

$$\hat{R}_t = \psi_1 \hat{\pi}_t + \epsilon_{R,t}. \tag{11}$$

- Substitute for $\hat{R}_t$ in Euler equation, replace $\hat{y}_t$ by $I\!\!E_{t-1}[\hat{y}_t] + \eta_t^y$ and $\hat{\pi}_t$ by $I\!\!E_{t-1}[\hat{\pi}_t] + \eta_t^\pi$

- and first focus on the evolution of the conditional expectations:

$$\Gamma_0 \underbrace{\begin{bmatrix} I\!\!E_t[\hat{y}_{t+1}] \\ I\!\!E_t[\hat{\pi}_{t+1}] \end{bmatrix}}_{s_t} = \Gamma_1 \underbrace{\begin{bmatrix} I\!\!E_{t-1}[\hat{y}_t] \\ I\!\!E_{t-1}[\hat{\pi}_t] \end{bmatrix}}_{s_{t-1}} + \Psi \epsilon_t + \Pi \underbrace{\begin{bmatrix} \hat{y}_t - I\!\!E_{t-1}[\hat{y}_t] \\ \hat{\pi}_t - I\!\!E_{t-1}[\hat{\pi}_t] \end{bmatrix}}_{\eta_t}, \tag{12}$$

where $\epsilon_t = \epsilon_{R,t}$.

Example

- The system matrices are

$$\Gamma_0 = \begin{bmatrix} 1 & 1/\tau \\ 0 & \beta \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 1 & \psi_1/\tau \\ -\kappa & 1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 1/\tau \\ 0 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 1 & \psi_1/\tau \\ -\kappa & 1 \end{bmatrix}.$$

Example

- Use method that is similar to Sims (2002) and has been extended to include sunspot equilibria by Lubik and Schorfheide (2003).

- We premultiply the above system by Γ_0 to solve for ξ_t:

$$s_t = \Gamma_1^* s_{t-1} + \Psi^* \epsilon_t + \Pi^* \eta_t. \tag{13}$$

- We proceed with a Jordan decomposition of $\Gamma_1^* = J \Lambda J^{-1}$. Define $w_t = J^{-1} s_t$ and write:

$$w_t = \Lambda w_{t-1} + J^{-1} \Psi^* \epsilon_t + J^{-1} \Pi^* \eta_t. \tag{14}$$

- If $I\!E_{t-1}[\epsilon_t] = 0$ then a solution of the LRE model is a function

$$\eta_t = \eta_1 \epsilon_t + \eta_2 \zeta_t, \tag{15}$$

where ζ_t is a vector of sunspot shocks such that w_t is stable.

Example

- We partition

$$w_t = \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix}$$

  such that the partitions of w_t and Λ conform and Λ_{22} contains all the explosive eigenvalues of Γ_1^* on its diagonal.

- We write the unstable block of the transformed system as

$$\begin{aligned} w_{2,t} &= \Lambda_2 w_{2,t-1} + [J^{-1}\Psi^*]_{2.}\epsilon_t + [J^{-1}\Pi^*]_{2.}\eta_t \qquad (16) \\ &= \Lambda_2 w_{2,t-1} + \widetilde{\Psi}\epsilon_t + \widetilde{\Pi}\eta_t \\ &= \Lambda_2 w_{2,t-1} + z_t. \end{aligned}$$

- For w_t to be non-explosive we need to choose η_t such that $z_t = 0$ for all t.

Example

- In principle we can distinguish three cases:

  1. uniqueness: there exists a unique mapping from the structural shocks into the expectation errors that leads to a stable law of motion for w_t;
  2. indeterminacy: there are multiple mappings from fundamental shocks and sunspot shocks into the expectation errors;
  3. non-existence: no stable rational expectations solution exists.

Example

- If $\psi_1 > 1$ both eigenvalues are unstable. The only stable solution is $\xi_t = 0$, obtained when

$$\Psi^* \epsilon_t + \Pi^* \eta_t = 0. \tag{17}$$

- Thus,

$$\eta_t = -\Pi^{*-1} \Psi^* \epsilon_t. \tag{18}$$

- It can be verified (using somewhat tedious algebra) that the the law of motion for output, inflation, and interest rates is of the form

$$\begin{bmatrix} y_t \\ \pi_t \\ R_t \end{bmatrix} = \frac{1}{\tau + \kappa\psi_1} \begin{bmatrix} -1 \\ -\kappa \\ \tau \end{bmatrix} \epsilon_{R,t} \tag{19}$$

Solving DSGE Models – Slightly More General

- Relax the assumption that $I\!E_{t-1}[\epsilon_t] = 0$...

- Again, we write the unstable block of the transformed system as

$$\begin{aligned} w_{2,t} &= \Lambda_2 w_{2,t-1} + [J^{-1}\Psi^*]_{2.}\epsilon_t + [J^{-1}\Pi^*]_{2.}\eta_t \qquad (20) \\ &= \Lambda_2 w_{2,t-1} + \widetilde{\Psi}\epsilon_t + \widetilde{\Pi}\eta_t \\ &= \Lambda_2 w_{2,t-1} + z_t. \end{aligned}$$

- Thus, we can write

$$w_{2,t} = \Lambda_2^{-1} w_{t+1} - \Lambda_2^{-1} z_{t+1} = -\sum_{j=1}^{\infty} (\Lambda_2^{-1})^j z_{t+j}. \qquad (21)$$

- Notice that

$$w_{2,t} = I\!E_t[w_{2,t}] = I\!E_{t+1}[W_{2,t}] \qquad (22)$$

- Hence,

$$w_{2,t} = \sum_{j=1}^{\infty} (\Lambda_2^{-1})^j I\!\!E_t[z_{t+j}] = \sum_{j=1}^{\infty} (\Lambda_2^{-1})^j I\!\!E_{t+1}[z_{t+j}] \tag{23}$$

- Since the η_t's are rational expectations errors, $I\!\!E_t[\eta_{t+j}] = 0$ for $j > 0$ and

$$w_{2,t} = \sum_{j=1}^{\infty} (\Lambda_2^{-1})^j \widetilde{\Psi} I\!\!E_t[\epsilon_{t+j}] = \Lambda_2^{-1} \widetilde{\Pi} \eta_{t+1} + \sum_{j=1}^{\infty} (\Lambda_2^{-1})^j \widetilde{\Psi} I\!\!E_{t+1}[\epsilon_{t+j}] \tag{24}$$

- Assuming that $\widetilde{\Pi}$ is invertible we can solve for η_{t+1}

$$\eta_{t+1} = \widetilde{\Pi}^{-1} \sum_{j=1}^{\infty} (\Lambda_2^{-1})^{j-1} \widetilde{\Psi} (I\!\!E_t[\epsilon_{t+j}] - I\!\!E_{t+1}[\epsilon_{t+j}]) \tag{25}$$

- It can now be easily verified that $I\!\!E_t[\eta_{t+1}] = 0$. If $I\!\!E_t[\epsilon_{t+j}] = 0$ for $j > 0$ then the condition simplifies to

$$\eta_{t+1} = -\widetilde{\Pi}^{-1} \widetilde{\Psi} \epsilon_{t+1} \tag{26}$$

which implies that $w_{2,t} = 0$ for all t, provided $w_{2,0} = 0$.

- Notice that once we have an expression for η_t in terms of the structural shock, we can substitute that expression into (7) to obtain a vector autoregressive law of motion for s_t.

- The general case is analyzed in Sims (2002). In particular, the assumption that Γ_0 is invertible is relaxed and the Jordan decomposition is replaced by a General Complex Schur decomposition.

Solving DSGE Models – Nonlinear Methods

- Second-order Perturbation Methods: Judd (1998), Collard and Juillard (2001), Jin and Judd (2002), Schmitt-Grohe and Uribe (2004), Kim, Kim, Schaumburg, Sims (2005), Swanson, Anderson, and Levin (2005).
- Judd's (1998) book covers various global approximation schemes.