An Application of Bayesian Analysis

Frank Schorfheide
Department of Economics, University of Pennsylvania

Application: Non-stationary Hours

- Based on Chang, Doh, and Schorfheide (2006): "Non-stationary Hours in a DSGE Model"

- Many researchers doubt that hours worked are stationary as we have observed apparent changes in labor-supply patterns over recent decades.

- We present a modified stochastic growth model in which hours worked have a stochastic trend, generated by a non-stationary labor supply shock.

- Based on output and hours data we compute posterior odds for four versions of the stochastic growth model obtained by using either a stationary or non-stationary labor supply shock.

Application: Non-stationary Hours

- We consider four versions of the model:
- In $\mathcal{M}_0$ and $\mathcal{M}_1$ firms can choose the employment level at the given wage rate without any adjustment cost.
- In $\mathcal{A}_0$ and $\mathcal{A}_1$, on the other hand, it is costly for firms to adjust the employment level.
- In $\mathcal{A}_0$ and $\mathcal{M}_0$ the labor supply shock is a stationary AR(1) process, whereas it is modelled as random walk in $\mathcal{A}_1$ and $\mathcal{M}_1$.

Application: Non-stationary Hours

- The representative household maximizes the expected discounted lifetime utility from consumption C_t and hours worked H_t:

$$E_t \left[\sum_{s=0}^{\infty} \beta^{t+s} \left(\ln C_{t+s} - \frac{(H_{t+s}/B_{t+s})^{1+1/\nu}}{1+1/\nu} \right) \right]. \tag{1}$$

- The log utility in consumption implies a constant long-run labor supply in response to a permanent change in technology. The short-run (Frisch) labor supply elasticity is ν. The labor supply shock is denoted by B_t.

- Per-period budget constraint faced by the household is

$$C_t + K_{t+1} - (1-\delta)K_t = W_t H_t + R_t K_t. \tag{2}$$

Application: Non-stationary Hours

- Firms rent capital, hire labor services, and produce final goods according to the following Cobb-Douglas technology:

$$Y_t = (A_t H_t)^\alpha K_t^{1-\alpha} \left(1 - \varphi \cdot \left(\frac{H_t}{H_{t-1}} - 1 \right)^2 \right). \tag{3}$$

- The stochastic process A_t represents the exogenous labor augmenting technical progress. The last term captures the cost of adjusting labor inputs: $\varphi \geq 0$.

- The firms maximize expected discounted future profits

$$\mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^{t+s} \lambda_{t+s} (Y_t - W_t H_t^d - R_t K_t^d) \right], \tag{4}$$

where λ_t is the marginal value of a unit consumption to a household, which is treated as exogenous to the firm.

Application: Non-stationary Hours

- In equilibrium $\lambda_t = 1/C_t$ and the goods, labor, and capital markets clear:

$$Y_t = C_t + K_{t+1} - (1-\delta)K_t, \qquad H_t^d = H_t, \quad \text{and} \quad K_t^d = K_t.$$

- We assume that the log production technology evolves according to a random walk with drift:

$$\ln A_t = \gamma + \ln A_{t-1} + \epsilon_{a,t}, \quad \epsilon_{a,t} \sim iid\mathcal{N}(0, \sigma_a^2). \tag{5}$$

The level of technology in period 0 is denoted by A_0.

Application: Non-stationary Hours

- In models $\mathcal{M}_0$ and $\mathcal{A}_0$, the labor supply shock follows a stationary AR(1) process:

$$\mathcal{M}_0: \quad \ln B_t = \rho_b \ln B_{t-1} + (1-\rho_b)\ln B_0 + \epsilon_{b,t}, \quad \epsilon_{b,t} \sim iid\mathcal{N}(0,\sigma_b^2), \tag{6}$$

where $0 \leq \rho_b < 1$ and $\ln B_0$ is the unconditional mean of $\ln B_t$.

- In model $\mathcal{M}_0$ and $\mathcal{A}_0$ the innovation $\epsilon_{b,t}$ only has a transitory effect. Alternatively, in models $\mathcal{M}_1$ and $\mathcal{A}_1$ the labor supply shock evolves according to a random walk:

$$\mathcal{M}_1: \quad \ln B_t = \ln B_{t-1} + \epsilon_{b,t}, \quad \epsilon_{b,t} \sim iid\mathcal{N}(0,\sigma_b^2) \tag{7}$$

and we use $\ln B_0$ to denote the initial level of $\ln B_t$.

Application: Non-stationary Hours

- It is well known that in models $\mathcal{M}_0$ and $\mathcal{A}_0$ hours are stationary and that output, consumption, and capital grow according to the technology process A_t. Hence, one can induce stationarity with the following transformation:

$$\mathcal{M}_0: \quad \widetilde{Y}_t = \frac{Y_t}{A_t}, \quad \widetilde{C}_t = \frac{C_t}{A_t}, \quad \widetilde{K}_{t+1} = \frac{K_{t+1}}{A_t}.$$

- In models $\mathcal{M}_1$ and $\mathcal{A}_1$, on the other hand, the labor supply shock B_t induces a stochastic trend into hours as well as output, consumption, and capital. To obtain a stationary equilibrium these variables have to be detrended according to:

$$\mathcal{M}_1: \quad \widetilde{H}_t = \frac{H_t}{B_t}, \quad \widetilde{Y}_t = \frac{Y_t}{A_t B_t}, \quad \widetilde{C}_t = \frac{C_t}{A_t B_t}, \quad \widetilde{K}_{t+1} = \frac{K_{t+1}}{A_t B_t}.$$

Application: Non-stationary Hours

- We fit the DSGE models to observations on the log level of real per capita output and hours worked, denoted by the 2×1 vector y_t.
- Let $\epsilon_t = [\epsilon_{a,t}, \epsilon_{b,t}]'$ and
- define the vector of structural model parameters as $\theta = [\alpha, \beta, \gamma, \delta, \nu, \ln A_0, \ln B_0, \rho_b, \sigma_a, \sigma_b]'$.
- It is well known that log-linearized DSGE models have a state space representation:

$$
\begin{aligned}
y_t &= \Gamma_0 + \Gamma_1 s_{1,t} + \Gamma_2 s_{2,t} + \Gamma_3 t && (8) \\
s_{1,t} &= \Phi_1 s_{1,t-1} + \Psi_1 \epsilon_t && (9) \\
s_{2,t} &= s_{2,t-1} + \Psi_2 \epsilon_t. && (10)
\end{aligned}
$$

Application: Non-stationary Hours

- The trend in (8) captures the effect of the drift in the random walk technology process A_t.
- Equation (9) represents the law of motion for the state variables of the detrended model,
- and (10) describes the evolution of trends: $s_{2,t} = \ln A_t - \gamma t$ in models $\mathcal{M}_0$ and $\mathcal{A}_0$ and $s_{2,t} = [\ln A_t - \gamma t, \ln B_t]'$ in $\mathcal{M}_1$ and $\mathcal{A}_1$.
- The Kalman filter can be used to compute the likelihood function $\mathcal{L}(\theta|Y^T)$ for the state space system (8) - (10).

Application: Non-stationary Hours

- To initialize the Kalman filter a distribution for the state vector in period $t = 0$ has to be specified.

- We factorize the initial distribution as $p(s_{1,0})p(s_{2,0})$ and set the first component equal to the unconditional distribution of $s_{1,t}$, whereas the second component, composed of the distribution of $\ln A_0$ (for $\mathcal{M}_0, \mathcal{A}_0$) and $[\ln A_0, \ln B_0]'$ (for $\mathcal{M}_1, \mathcal{A}_1$), respectively, is absorbed into the specification of our prior $p(\theta)$.

Application: Non-stationary Hours

- According to Bayes Theorem the posterior distribution of θ is given by

$$p(\theta|Y^T) = \mathcal{L}(\theta|Y^T)p(\theta)/p(Y^T). \tag{11}$$

- The fit of models can be assessed based on the marginal data densities:

$$p(Y^T) = \int \mathcal{L}(\theta|Y^T)p(\theta)d\theta. \tag{12}$$

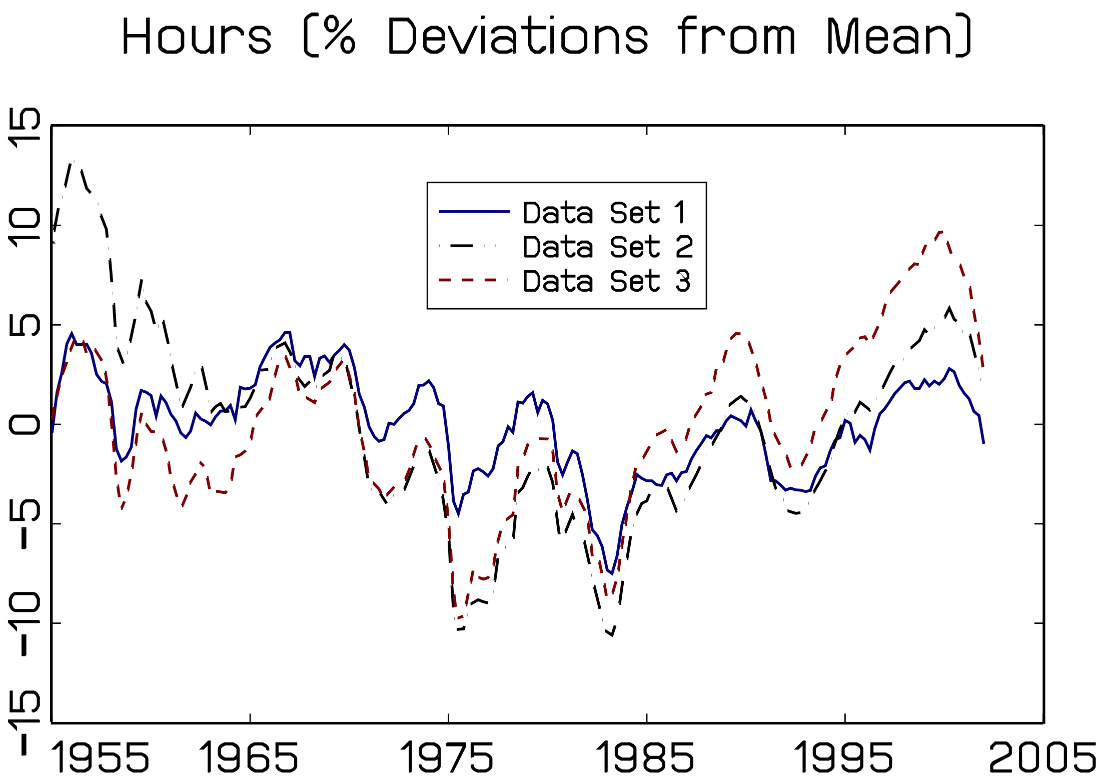
- Moreover, we will conduct posterior predictive checks.

Application: Non-stationary Hours

- Paper uses three different data sets comprised of quarterly U.S. real per capita GDP and hours worked from 1954:Q2 to 2001:Q4.

- For Data Set 1 we use real GDP from the DRI-Global Insight database (GDPQ) and divide it by population of age 20 or older (PM20+PF20). Hours worked is measured as average weekly hours of all people in the non-farm business sector compiled by the Bureau of Labor Statistics (EEU00500005). We multiply the hours series by the employment ratio, which is the number of people employed (LHEM, DRI-Global Insight) divided by population (PM20+PF20).

Figure 1: Hours Worked Data



Application: Non-stationary Hours

- The observations from 1954:Q2 to 1958:Q4 are treated as pre-sample to quantify prior distributions.

- Since we are comparing the fit of the DSGE model specifications to that of a VAR with 4 lags, we reserve the observations from 1959:Q1 to 1959:Q4 for the initialization of lags.

- Since the VAR likelihood function is conditional on the observations from the year 1959, we adjust the DSGE model likelihood function accordingly.

Application: Non-stationary Hours

- We assume all parameters to be *a priori* independent.
- By and large, the prior means are chosen based on a pre-sample of observations from 1954:Q2 to 1958:Q4.
- The prior mean of the labor share α is 0.66 and that for the quarter-to-quarter growth rate of productivity, γ, is 0.5%.
- The prior for β is centered at 0.995. Combined with the prior mean of γ, this corresponds to an annualized real return of about 4%.
- The depreciation rate δ lies between 1.8% and 3.3% per quarter.
- The 90% probability interval for the Frisch labor supply elasticity ν ranges from 0.3 to 1.8.

- We specify a prior for the adjustment cost parameter φ as follows.
  - In order to recruit labor ΔH, firms can either search for workers, incurring adjustment costs $\varphi(\frac{\Delta H}{H})^2 Y$, or pay head hunters for finding workers.
  - In the latter case the head hunters service fee is $\zeta W \Delta H$ where ζ is the fraction of the salary of the job to fill.
  - It is known that the head hunters tend to charge about 1/3 to 2/3 of quarterly earnings of a worker (i.e., $\zeta = 1/3$ to $2/3$).
  - At the margin, the recruiting costs should be the same: $\varphi(\frac{\Delta H}{H})^2 Y = \zeta W \Delta H$.
  - With the labor share of 1/3 ($=\frac{WH}{Y}$) for a size of one percent increase of employment, $\frac{\Delta H}{H} = 1\%$, we obtain a range of 22 to 44 for φ.
  - We use a fairly diffuse prior distribution that is centered at 33 and has a standard deviation of 15.

- The presence of adjustment costs dampens the effect of technology and labor supply shocks on output and hours worked.

- In order to guarantee that the adjustment cost specifications have *a priori* similar implications for the volatility of the endogenous variable as $\mathcal{M}_0$ and $\mathcal{M}_1$ we use slightly different priors for the standard deviations of the structural shocks.

- Under $\mathcal{M}_0$ and $\mathcal{M}_1$ the priors for σ_a and σ_b are centered at 0.010, whereas under $\mathcal{A}_0$ and $\mathcal{A}_1$ they are centered at 0.015.

Application: Non-stationary Hours

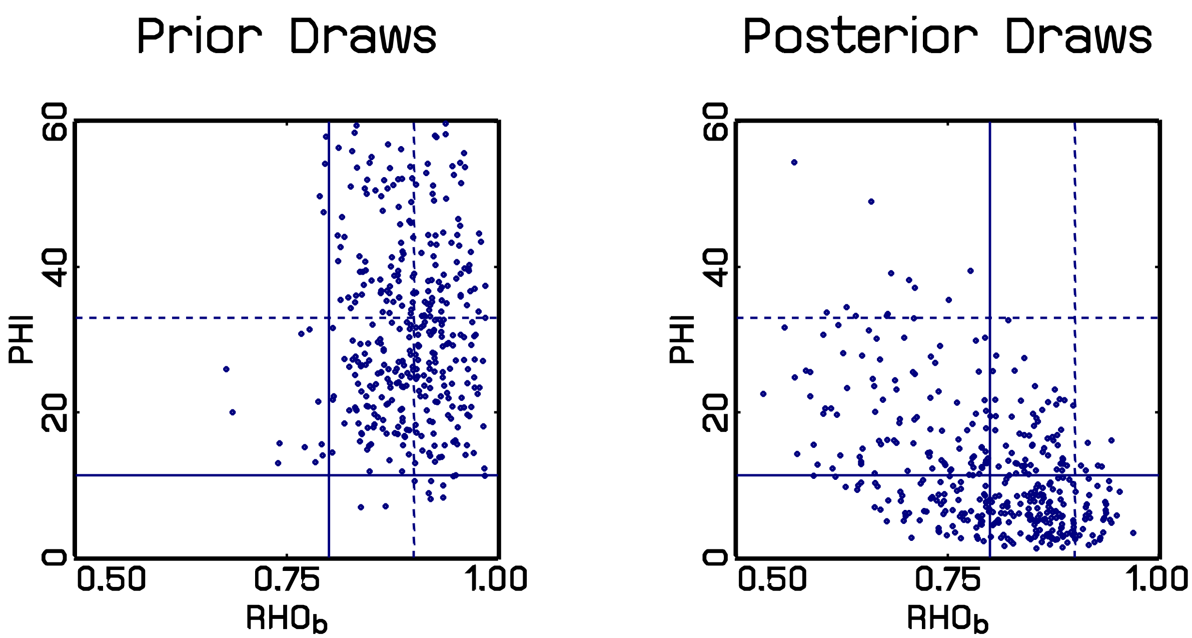
- For $\mathcal{M}_0$ and $\mathcal{A}_0$ the prior mean of $\ln B_0$ is constructed by matching average hours worked over the pre-sample period with the steady state level of hours worked $\widetilde{H}^*$, evaluated at the prior mean values of the remaining structural parameters.

- For $\mathcal{M}_1$ and $\mathcal{A}_1$ the prior mean of $\ln B_0$ is obtained by equating hours worked in 1958:Q4 with the steady state level $B_0\widetilde{H}^*$. Similarly, we select the prior mean of $\ln A_0$ by matching $A_0\widetilde{Y}^*$ and $A_0B_0\widetilde{Y}^*$, respectively, with the level of output in 1958:Q4.

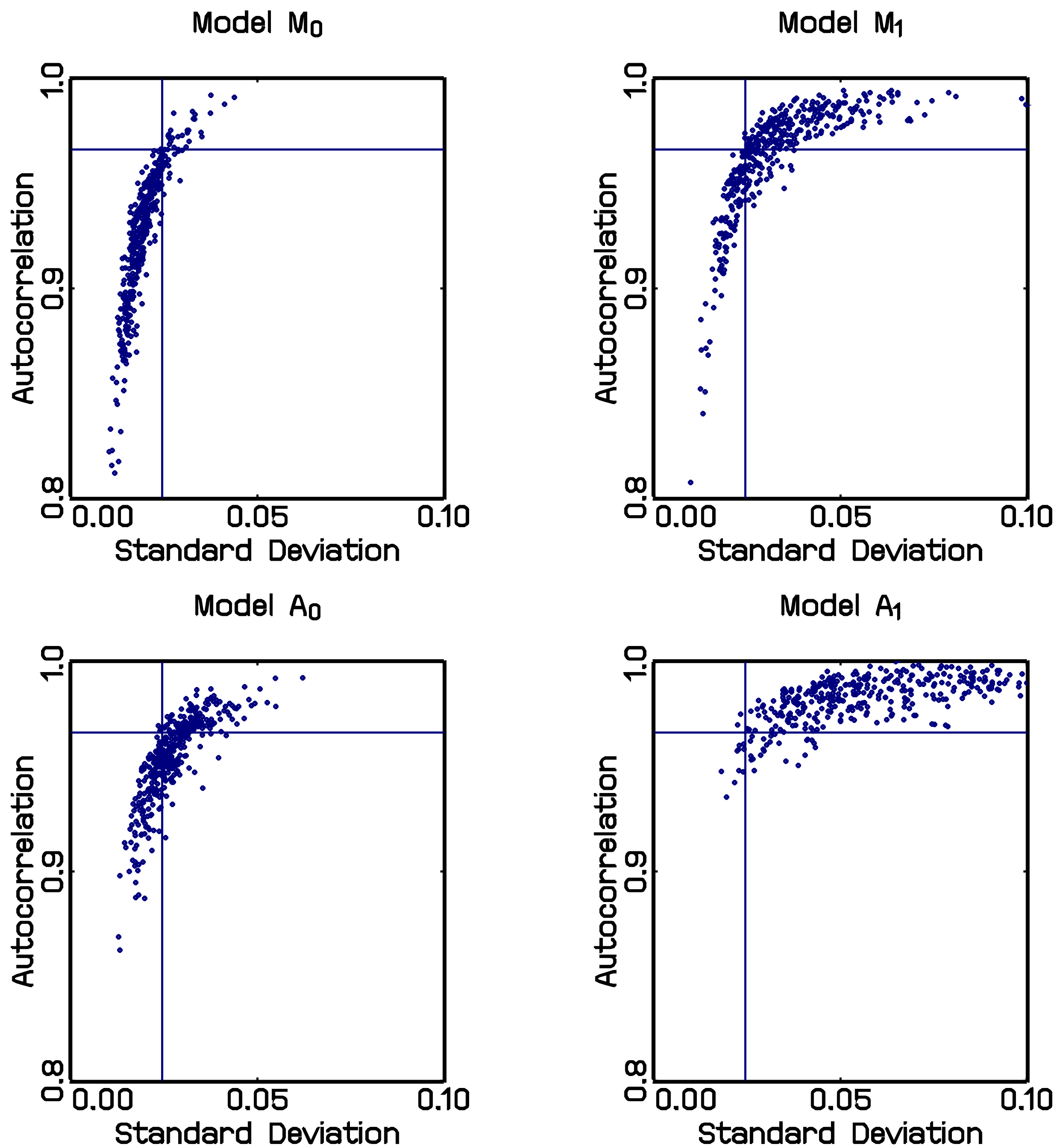
- The prior standard deviations for $\ln A_0$ and $\ln B_0$ are 0.2. Finally, for $\mathcal{M}_0$ and $\mathcal{A}_0$ the 90% probability interval for the autoregressive parameter ρ_b ranges from 0.825 to 0.977, implying a fairly persistent labor supply process.

Parameter	Domain	Density	Data Set	Model	Para (1)	Para (2)
α	$[0,1)$	Beta	all	all	0.660	0.020
β	$[0,1)$	Beta	all	all	0.995	0.002
γ	$\mathbb{R}$	Normal	all	all	0.005	0.005
δ	$[0,1)$	Beta	all	all	0.025	0.005
ν	$\mathbb{R}^+$	Gamma	all	all	1.000	0.500
ρ_b	$[0,1)$	Beta	all	$\mathcal{M}_0$, $\mathcal{A}_0$	0.900	0.050
σ_a	$\mathbb{R}^+$	InvGamma	all	$\mathcal{M}_0$, $\mathcal{M}_1$	0.010	1.000
			all	$\mathcal{M}_1$, $\mathcal{A}_1$	0.015	1.000
σ_b	$\mathbb{R}^+$	InvGamma	all	$\mathcal{M}_0$, $\mathcal{M}_1$	0.010	1.000
			all	$\mathcal{M}_1$, $\mathcal{A}_1$	0.015	1.000
$\ln A_0$	$\mathbb{R}$	Normal	1	$\mathcal{M}_0$, $\mathcal{A}_0$	5.647	0.200
			1	$\mathcal{M}_1$, $\mathcal{A}_1$	5.674	0.200
$\ln B_0$	$\mathbb{R}$	Normal	1	$\mathcal{M}_0$, $\mathcal{A}_0$	3.236	0.200
			1	$\mathcal{M}_1$, $\mathcal{A}_1$	3.209	0.200
φ	$\mathbb{R}^+$	Gamma	all	$\mathcal{A}_0$, $\mathcal{A}_1$	33.00	15.00

Parameter	Domain	Density	Data Set	Model	Para (1)	Para (2)
			Alternative Prior P1			
$\ln B_0$	$\mathbb{R}$	Normal	1	$\mathcal{M}_1$	3.209	2.000
			2	$\mathcal{M}_1$	6.405	2.000
			3	$\mathcal{M}_1$	6.309	2.000
			Alternative Prior P2			
$\ln B_0$	$\mathbb{R}$	Normal	1	$\mathcal{M}_1$	3.209	0.020
			2	$\mathcal{M}_1$	6.405	0.020
			3	$\mathcal{M}_1$	6.309	0.020
			Alternative Prior P3			
ρ_b	$[0, 1)$	Beta	all	$\mathcal{M}_0$	0.980	0.005
			Alternative Prior P4			
ρ_b	$[0, 1)$	Beta	all	$\mathcal{M}_0$	0.800	0.100


Prior Draws
Posterior Draws
PHI
RHOb
0.50
0.75
1.00
0
20
40
60

Data Set	Prior	$\mathcal{M}_0$	$\mathcal{M}_1$	$\mathcal{A}_0$	$\mathcal{A}_1$	VAR(4)
1	B	1176.33	1178.45	1182.10	1180.21	1180.49
	P1		1176.81			
	P2		1178.61			
	P3	1177.64				
	P4	1174.85				


Model M_0
Model M_1
Model A_0
Model A_1
Autocorrelation
Standard Deviation
1.0
0.9
0.8
0.00
0.05
0.10