Marginal Likelihoods and Posterior Odds

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Posterior Odds and Marginal Data Densities

- Posterior model probabilities can be computed as follows:

$$\pi_{i,T} = \frac{\pi_{i,0} p(Y|\mathcal{M}_i)}{\sum_j \pi_{j,0} p(Y|\mathcal{M}_j)}, \quad j = 1, \ldots, 4, \tag{1}$$

- where

$$p(Y|\mathcal{M}) = \int \mathcal{L}(\theta|Y, \mathcal{M}) p(\theta|\mathcal{M}) d\theta \tag{2}$$

- Posterior odds and Bayes Factor

$$\frac{\pi_{1,T}}{\pi_{2,T}} = \underbrace{\frac{\pi_{1,0}}{\pi_{2,0}}}_{Prior\,Odds} \times \underbrace{\frac{p(Y|\mathcal{M}_1)}{p(Y|\mathcal{M}_2)}}_{Bayes\,Factor} \tag{3}$$

Example: Linear Regression

- Simple example: compare

$$\mathcal{M}_0 \; : \; y_t = x_t^{(0)'}\theta^{(0)} + u_t^{(0)} \tag{4}$$

$$\mathcal{M}_1 \; : \; y_t = x_t^{(1)'}\theta^{(1)} + u_t^{(1)} \tag{5}$$

- Prior probabilities: $\pi_{i,0}$.

- Posterior probabilities:

$$\pi_{i,T} = \frac{\pi_{i,0}p(Y^T|\mathcal{M}_i)}{\sum_{i=0,1}\pi_{i,0}p(Y^T|\mathcal{M}_i)}. \tag{6}$$

  where marginal data density is

$$p(Y^T|\mathcal{M}_i) = \int \mathcal{L}(\theta^{(i)}|Y^T,\mathcal{M}_i)p(\theta^{(i)}|\mathcal{M}_i)d\theta^{(i)} \tag{7}$$

Example: Linear Regression

- Here calculation is relatively simple:

$$p(Y|X) = \frac{\mathcal{L}(\theta|Y,X)p(\theta)}{p(\theta|Y,X)}. \tag{8}$$

- Since, we previously showed that the posterior $p(\theta|Y,X)$ is multivariate normal all the terms on the right-hand-side are known:

$$\begin{aligned} p(Y|X) &= \frac{(2\pi)^{-T/2}(2\pi)^{-k/2}\tau^{-k}\exp\left\{-\frac{1}{2}[(Y-X\theta)'(Y-X\theta)+\theta'\theta/\tau^2]\right\}}{(2\pi)^{-k/2}|\tilde{V}|^{-1/2}\exp\left\{-\frac{1}{2}[(\theta-\tilde{\theta})'\tilde{V}^{-1}(\theta-\tilde{\theta})]\right\}} \\ &= (2\pi)^{-T/2}\tau^{-k}|X'X+\tau^{-2}\mathcal{I}|^{-1/2} \\ &\quad \times \exp\left\{-\frac{1}{2}[Y'Y-Y'X(X'X+\tau^{-2}\mathcal{I})^{-1}X'Y]\right\}. \end{aligned} \tag{9}$$

using the definition of $\tilde{\theta}$ and $\tilde{V}$:

$$\tilde{\theta}_T = (X'X+\tau^{-2}\mathcal{I})^{-1}X'Y, \quad \tilde{V}_T = (X'X+\tau^{-2}\mathcal{I})^{-1}.$$

Example: Linear Regression

- Schwarz Criterion: The terms of the marginal data density that asymptotically dominate are

$$\begin{aligned}\ln p(Y|X) &= -\frac{T}{2}\ln(2\pi) - \frac{1}{2}(Y'Y - Y'X(X'X)^{-1}X'Y) - \frac{k}{2}\ln T + small \\ &= \underbrace{\ln p(Y|X,\hat{\theta}_{mle})}_{\text{max likelihood}} - \underbrace{\frac{k}{2}\ln T}_{\text{penalty}} + small \qquad (10)\end{aligned}$$

- Notice that

$$\begin{aligned}\ln|X'X + \tau^{-2}\mathcal{I}|^{-1/2} &= -\frac{1}{2}\ln\left|T\left(\frac{1}{T}X'X + \frac{1}{T\tau^2}\mathcal{I}\right)\right| \qquad (11) \\ &= \underbrace{-\frac{k}{2}\ln T}_{O(\ln(T))} - \underbrace{\frac{1}{2}\ln\left|\frac{1}{T}X'X + \frac{1}{T\tau^2}\mathcal{I}\right|}_{O_p(1)}\end{aligned}$$

Consistency

- If data are generated from model $\mathcal{M}_i$ then as $T \longrightarrow \infty$ the posterior probability of model $\mathcal{M}_i$ converges to one for almost all data sets.

  If the models are nested, then the posterior prob of the smaller model will converge to one (because the penalty is smaller).

- If data are not generated from any of the models under consideration, then, roughly speaking, the posterior probability of the model that is closest in the Kullback-Leibler sense to the "truth" converges to one.

  What happens if there are ties? Is this a very useful result?

Example: Linear Regression

- Suppose we compare: $\mathcal{M}_0$ $y_t = u_t$ versus $\mathcal{M}_1$ $y_t = x_t'\theta + u_t$, $\theta \sim \mathcal{N}(0, \tau^2\mathcal{I})$.

- Under $\mathcal{M}_0$:

$$\ln p(Y|X) = -\frac{T}{2}\ln(2\pi) - \frac{1}{2}Y'Y$$

  whereas under $\mathcal{M}_1$:

$$\ln p(Y|X) = -\frac{T}{2}\ln(2\pi) - \frac{1}{2}(Y'Y - Y'X(X'X + \tau^{-2}\mathcal{I})^{-1}X'Y) - \frac{k}{2}\ln T + small$$

Example: Linear Regression

- Assume that data were generated from the model $y_t = x_t'\theta_0 + u_t$.

$$\begin{aligned} & Y'X(X'X+\tau^{-2})^{-1}X'Y \\ &= \theta_0'X'X(X'X+\tau^{-2})^{-1}X'X\theta_0 + U'X(X'X+\tau^{-2})^{-1}X'U \\ &\quad +U'X(X'X+\tau^{-2})^{-1}X'X\theta_0 + \theta_0'X(X'X+\tau^{-2})^{-1}X'U \\ &= T\theta_0'\left(\frac{1}{T}\sum x_tx_t'\right)^{-1}\theta_0 + \sqrt{T}2\left(\frac{1}{\sqrt{T}}\sum x_tu_t\right)'\theta_0 \\ &\quad +\left(\frac{1}{\sqrt{T}}\sum x_tu_t\right)'\left(\frac{1}{T}\sum x_tx_t'\right)^{-1}\left(\frac{1}{\sqrt{T}}\sum x_tu_t\right) + O_p(1). \end{aligned} \tag{12}$$

- If $\theta_0 = 0$ then

$$\ln\left[\frac{p(Y|X,\mathcal{M}_0)}{p(Y|X,\mathcal{M}_1)}\right] = \frac{k}{2}\ln T + small \longrightarrow +\infty. \tag{13}$$

- If $\theta_0 \neq 0$ then

$$\ln\left[\frac{p(Y|X,\mathcal{M}_0)}{p(Y|X,\mathcal{M}_1)}\right] = -\frac{T}{2}\theta_0'\left(\frac{1}{T}\sum x_tx_t'\right)^{-1}\theta_0 + small \longrightarrow -\infty. \tag{14}$$

Finite-Sample Challenges: Lindley's Paradox

- Test $H_0 : \theta = 0$:

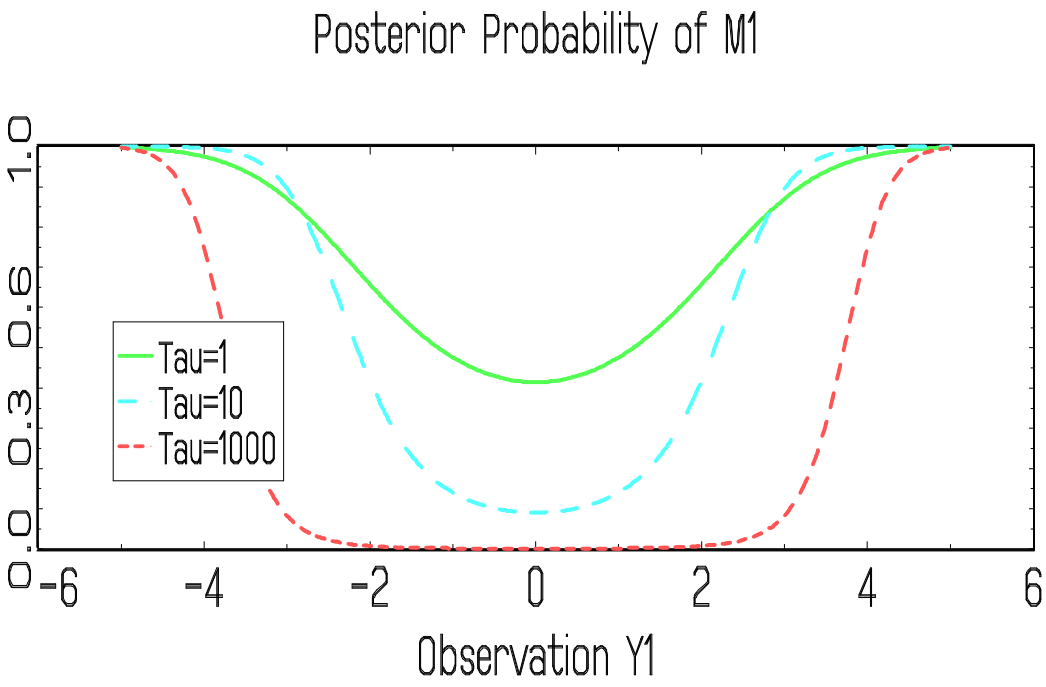
$$\ln\left[\frac{p(Y|X,\mathcal{M}_0)}{p(Y|X,\mathcal{M}_1)}\right] = \tau^k|X'X+\tau^{-2}|^{1/2}\exp\left\{-\frac{1}{2}[Y'X(X'X+\tau^{-2}\mathcal{I})^{-1}X'Y]\right\} \quad (15)$$

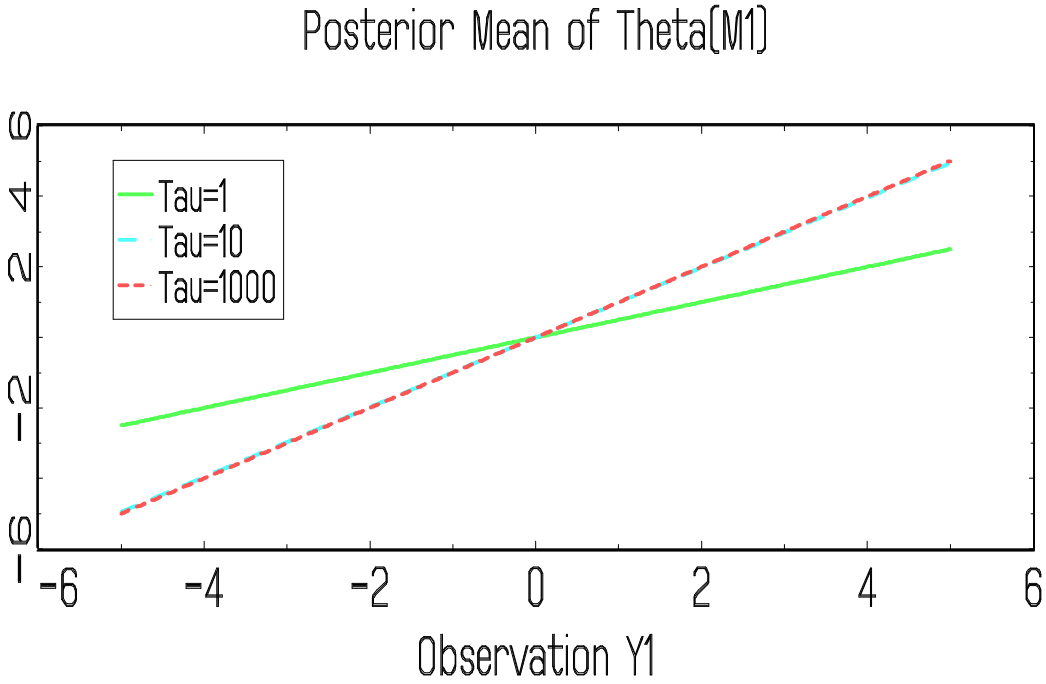
- Lindley's Paradox: suppose Y is fixed, as prior on alternative becomes more diffuse $(\tau \longrightarrow \infty)$

$$\ln\left[\frac{p(Y|X,\mathcal{M}_0)}{p(Y|X,\mathcal{M}_1)}\right] \longrightarrow \infty \quad (16)$$

  regardless of Y.

- Important for non-nested model comparisons. Changing prior variance can have small effect on posterior distribution of parameters yet large effect on posterior model probabilities.


Posterior Probability of M1
1.0
0.6
0.3
0.0
Tau=1
Tau=10
Tau=1000
-6
-4
-2
0
2
4
6
Observation Y1


Posterior Mean of Theta(M1)
6
4
2
-2
-6
Tau=1
Tau=10
Tau=1000
-6
-4
-2
0
2
4
6
Observation Y1

Example: Model Selection vs Likelihood Ratio Test

- Comparison of Bayesian test to LR test.
- In the regression example: $(H_0 : \theta = 0)$

$$LR = 2 \ln \left[\frac{p(Y|X, \hat{\theta}_{mle})}{p(Y|X, \theta = 0)} \right] \tag{17}$$

$$= Y'X(X'X)^{-1}X'Y. \tag{18}$$

- If H_0 is true, then

$$LR = \left(\frac{1}{\sqrt{T}} \sum x_t u_t \right)' \left(\frac{1}{T} \sum x_t x_t' \right)^{-1} \left(\frac{1}{\sqrt{T}} \sum x_t u_t \right) \Longrightarrow \chi^2_k \tag{19}$$

Example: Model Selection vs. Likelihood Ratio Tests

- Frequentist decision rule: accept / don't reject $\theta = 0$ if

$$Y'X(X'X)^{-1}X'Y < \chi^2_{k,crit} \tag{20}$$

- Bayesian decision rule: accept $\theta = 0$ if

$$Y'X(X'X)^{-1}X'Y < k \ln T + small \tag{21}$$

- Note: the implied Bayesian critical value tends to infinity at logarithmic rate. Consequently, the size of the test converges to zero asymptotically and the Type 1 error vanishes.

In General...

- Laplace approximation ($\tilde{\theta}$ is mode, $\tilde{\Sigma}$ is inv Hessian)

$$\hat{p}(Y|\mathcal{M}) = (2\pi)^{d/2} \underbrace{p(Y|\tilde{\theta},\mathcal{M})p(\tilde{\theta}|\mathcal{M})}_{In-sample\,Fit} \times \underbrace{|\tilde{\Sigma}|^{1/2}}_{Dimensionality\,Penalty} \tag{22}$$

- In "regular" models $T \cdot \tilde{\Sigma} = O_p(1)$. Thus,

$$\frac{1}{2}\ln|\tilde{\Sigma}| = -\frac{d}{2}\ln T \tag{23}$$

  The larger the dimensionality d, the larger the penalty.

- $\ln p(Y|\mathcal{M})$ can be interpreted as predictive score (Good, 1952)

$$\sum_{t=1}^{T} \ln p(y_t|Y^{t-1},\mathcal{M}) = \sum_{t=1}^{T} \ln\left[\int p(y_t|Y^{t-1},\theta,\mathcal{M})p(\theta|Y^{t-1},\mathcal{M})d\theta\right], \tag{24}$$

  Model comparison based on posterior odds captures the relative one-step-ahead predictive performance.

Numerical Approximations

- Naive approach: draw $\theta^{(s)}$ from prior $p(\theta)$ and use

$$\ln p(Y) \approx \frac{1}{n_{sim}} \sum_{s=1}^{n_{sim}} p(Y|\theta^{(s)})$$

- Why does this not work?

- Refinement:

$$\ln p(Y) = \prod_{t=1}^{T} p(y_t|Y^{t-1}) \approx \prod_{t=1}^{T} \frac{1}{n_{sim}} \sum_{s=1}^{n_{sim}} p(y_t|Y^{t-1}, \theta_{t-1}^{(s)}),$$

  where $\theta_{t-1}^{(s)}$) is drawn from $p(\theta|Y^{t-1})$.

- We will now discuss Geweke's modified harmonic mean estimator and the Chib and Jeliazkov method.

Numerical Approximations: Harmonic Mean

- Harmonic mean estimators are based on the following identity

$$\frac{1}{p(Y)} = \int \frac{f(\theta)}{\mathcal{L}(\theta|Y)p(\theta)} p(\theta|Y)d\theta, \tag{25}$$

  where $\int f(\theta)d\theta = 1$.

- Conditional on the choice of $f(\theta)$ an obvious estimator is

$$\hat{p}_G(Y) = \left[\frac{1}{n_{sim}} \sum_{s=1}^{n_{sim}} \frac{f(\theta^{(s)})}{\mathcal{L}(\theta^{(s)}|Y)p(\theta^{(s)})}\right]^{-1}, \tag{26}$$

  where $\theta^{(s)}$ is drawn from the posterior $p(\theta|Y)$.

- Geweke (1999):

$$\begin{aligned} f(\theta) &= \tau^{-1}(2\pi)^{-d/2}|V_\theta|^{-1/2} \exp\left[-0.5(\theta-\bar{\theta})'V_\theta^{-1}(\theta-\bar{\theta})\right] \\ &\quad \times \left\{(\theta-\bar{\theta})'V_\theta^{-1}(\theta-\bar{\theta}) \leq F^{-1}_{\chi^2_d}(\tau)\right\}. \end{aligned} \tag{27}$$

Numerical Approximations: Chib and Jeliazkov (I)

- Rewrite Bayes Theorem:

$$p(Y) = \frac{\mathcal{L}(\theta|Y)p(\theta)}{p(\theta|Y)}. \tag{28}$$

- Thus,

$$\hat{p}_{CS}(Y) = \frac{\mathcal{L}(\tilde{\theta}|Y)p(\tilde{\theta})}{\hat{p}(\tilde{\theta}|Y)}, \tag{29}$$

  where we replaced the generic θ in (28) by the posterior mode $\tilde{\theta}$.

- Within the RWM Algorithm denote the probability of moving from θ to ϑ by

$$\alpha(\theta, \vartheta|Y) = \min\ \{1, r(\theta, \vartheta|Y)\}, \tag{30}$$

  where $r(\theta, \vartheta|Y)$ was in the description of the algorithm. Moreover, let $q(\theta, \tilde{\theta}|Y)$ be the proposal density for the transition from θ to $\tilde{\theta}$.

Numerical Approximations: Chib and Jeliazkov (II)

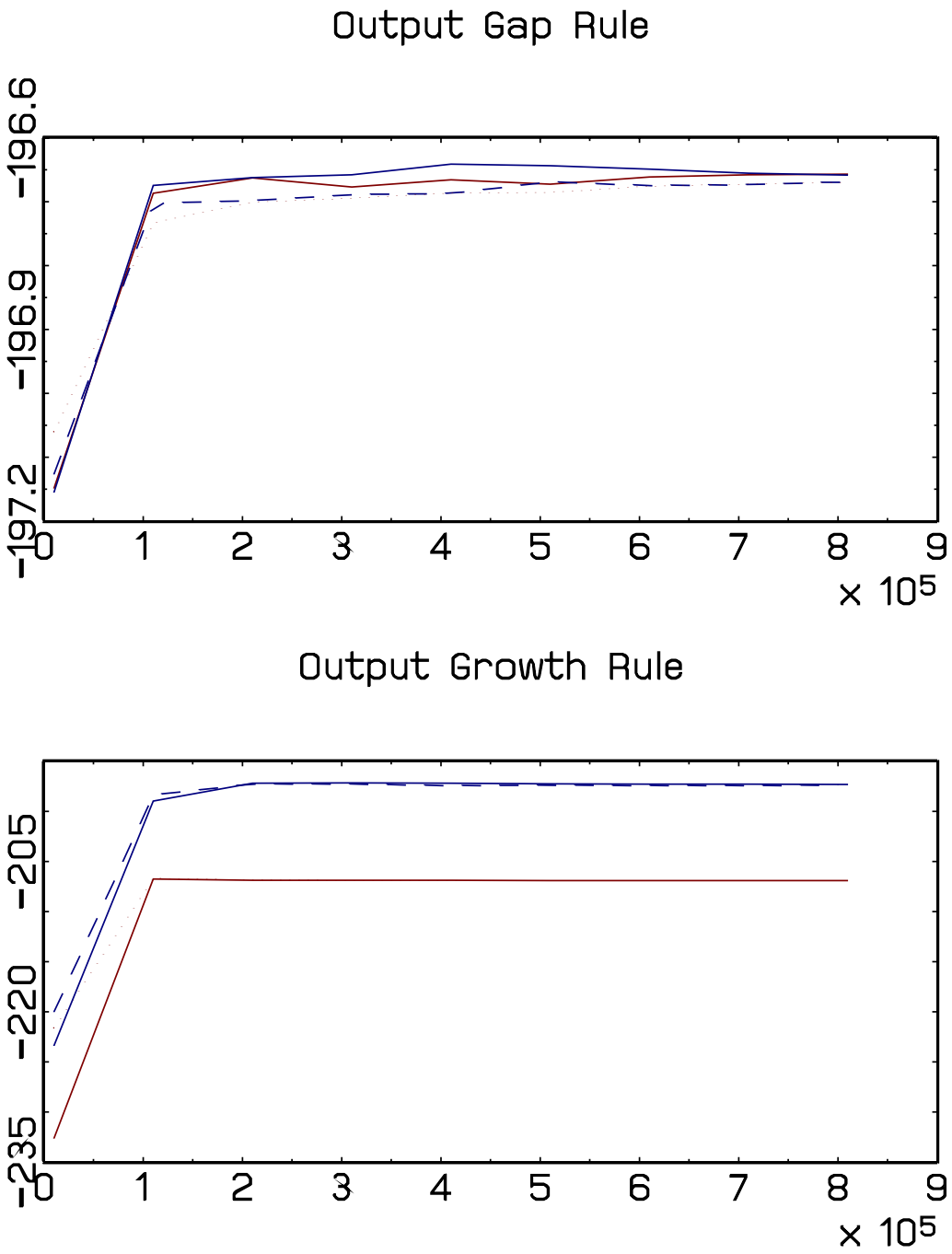
- Then the posterior density at the mode can be approximated as follows

$$\hat{p}(\tilde{\theta}|Y) = \frac{\frac{1}{n_{sim}} \sum_{s=1}^{n_{sim}} \alpha(\theta^{(s)}, \tilde{\theta}|Y) q(\theta^{(s)}, \tilde{\theta}|Y)}{J^{-1} \sum_{j=1}^{J} \alpha(\tilde{\theta}, \theta^{(j)}|Y)}, \tag{31}$$

  where $\{\theta^{(s)}\}$ are sampled draws from the posterior distribution with the RWM Algorithm and $\{\theta^{(j)}\}$ are draws from $q(\tilde{\theta}, \theta|Y)$ given the fixed posterior mode value $\tilde{\theta}$.

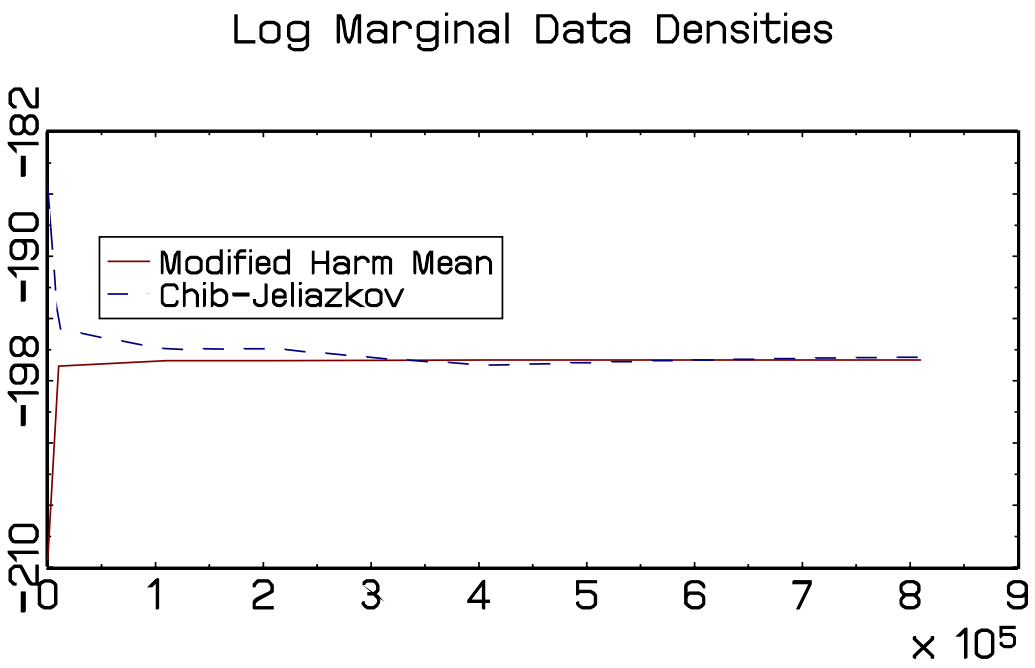
(insert figures here)

Figure 9: Log Marginal Data Densities from Multiple Chains



Notes: Output gap rule specification (top panel) and output growth rule specification (bottom panel), Data Sets 1-$\mathcal{M}_1$ and 1-$\mathcal{M}_2$, respectively. For each Markov chain, log marginal data densities are computed recursively with Geweke's modified harmonic mean estimator and plotted as a function of the number of draws.

Figure 10: Log Marginal Data Densities – Geweke vs. Chib-Jeliazkov



Notes: Output gap rule specification $\mathcal{M}_1$, Data Set 1-$\mathcal{M}_1$. Log marginal data densities are computed recursively with Geweke's modified harmonic mean estimator as well as the Chib-Jeliazkov estimator and plotted as a function of the number of draws.

Bayes Factors / Posterior Odds: Example

- Two alternative specifications: $\mathcal{M}_3$ prices are nearly flexible ($\kappa = 5$); $\mathcal{M}_4$ central bank does not respond to output ($\psi_2 = 0$).
- Marginal data densities are -196.7 for $\mathcal{M}_1$, -245.6 for $\mathcal{M}_3$, and -201.9 for $\mathcal{M}_4$.
- Bayes factors:
  - $\mathcal{M}_1$ versus $\mathcal{M}_3$ is approximately e^{49};
  - $\mathcal{M}_1$ versus $\mathcal{M}_4$ is approximately e^{4}.