Structural VARs

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(Lack of) Identification

- So far, we considered reduced form VARs, say,

$$y_t = \Phi_1 y_{t-1} + u_t, \quad I\!\!E[u_t u_t'] = \Sigma \tag{1}$$

- Error terms u_t have the interpretation of one-step ahead forecast errors.
- If the eigenvalues of Φ_1 are inside the unit-circle then y_t has the following moving-average (MA) representation in terms of u_t:

$$y_t = (I - \Phi_1 L)^{-1} u_t = \sum_{j=0}^{\infty} \Phi_1^j u_{t-j} = \sum_{j=0}^{\infty} C_j u_{t-j} \tag{2}$$

- DSGE models suggest that the one-step ahead forecast errors are functions of some fundamental shocks, such as technology shocks, preference shocks, or monetary policy shocks.

(Lack of) Identification

- Let ϵ_t a vector of such fundamental shocks and assume that $I\!E[\epsilon_t \epsilon_t'] = \mathcal{I}$. Moreover, assume that

$$u_t = \Phi_\epsilon \epsilon_t. \tag{3}$$

- Then we can express the VAR in structural form as follows

$$\begin{aligned} y_t &= \Phi_1 y_{t-1} + \Phi_\epsilon \epsilon_t \\ \Phi_\epsilon^{-1} y_t &= \Phi_\epsilon^{-1} \Phi_1 y_{t-1} + \epsilon_t \end{aligned} \tag{4}$$

- The moving-average representation of y_t in terms of the structural shocks is given by

$$y_t = \sum_{j=0}^{\infty} \Phi_1^j \Phi_\epsilon \epsilon_{t-j} = \sum_{j=0}^{\infty} C_j \Phi_\epsilon \epsilon_{t-j}. \tag{5}$$

(Lack of) Identification

- For (1) and (4) the matrix Φ_ϵ has to satisfy the restriction

$$\Phi_\epsilon \Phi'_\epsilon = \Sigma \tag{6}$$

  Notice that the matrix Φ_ϵ has n^2 elements.

- The covariance relationship, unfortunately, generates only $n(n+1)/2$ restrictions and does not uniquely determine Φ_ϵ.

- This creates an identification problem since all we can estimate from the data is Φ_1 and Σ.

(Lack of) Identification

- In order to make statements about the propagation of structural shocks ϵ_t we have to make further assumptions. The papers (see course outline) by Cochrane (1994), Christiano and Eichenbaum (1999), and Stock and Watson (2001) survey such identifying assumptions. A cynical view of this literature is the following:

  1. Propose an identification scheme, that determines all elements of Φ_ϵ.
  2. Compute impulse response functions.
  3. If impulse response functions are plausible, then stop; else, declare a "puzzle" and return to 1.

(Lack of) Identification

- Here are some famous "puzzles:"

  1. "Liquidity Puzzle:" When identifying monetary policy shocks as surprise changes in the stock of money one often finds that interest rates fall when the money stock is lowered.

  2. "Price Puzzle:" When identifying monetary policy shocks as surprise changes in the Federal Funds Rate, one often finds that prices fall after a drop in interest rates.

- These "puzzles" are typically resolved by considering more elaborate identification schemes.

Identification Schemes

- We begin by decomposing the covariance matrix into the product of lower triangular matrices (Cholesky Decomposition):

$$\Sigma = AA', \tag{7}$$

where A is lower triangular. If Σ is non-singular the decomposition is unique.

- Let Ω be an orthonormal matrix, meaning that $\Omega\Omega' = \Omega'\Omega = \mathcal{I}$.

- We can characterize the relationship between the reduced form and the structural shocks as follows

$$u_t = A\Omega\epsilon_t \tag{8}$$

- Notice that

$$I\!E[u_t u_t'] = I\!E[A\Omega\epsilon_t\epsilon_t'\Omega' A'] = A\Omega I\!E[\epsilon_t\epsilon_t']\Omega' A' = A\Omega\Omega' A' = AA' = \Sigma. \tag{9}$$

Identification Schemes

- In general, it is quite tedious to characterize the space of orthonormal matrices. Let's try for $n = 2$:

$$\Omega(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \tag{10}$$

where $\varphi \in (-\pi, \pi]$.

- Notice that, for instance,

$$\Omega(\pi/2) = -\Omega(-\pi/2) \tag{11}$$

which means that only the signs of the impulse responses change but not the shape.

- Identification schemes impose restrictions on φ.

Short-run Restrictions

- Suppose that

$$y_t = \begin{bmatrix} \text{Fed Funds Rate} \\ \text{Output Growth} \end{bmatrix}, \quad \epsilon_t = \begin{bmatrix} \epsilon_{R,t} \\ \epsilon_{z,t} \end{bmatrix} = \begin{bmatrix} \text{Monetary Policy Shock} \\ \text{Technology Shock} \end{bmatrix}.$$

Moreover, we assume that the central bank does not react contemporaneously to technology shocks because data on aggregate output only become available with a one-quarter lag.

This assumption can be formalized through $\varphi = 0$. Then

$$u_t = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \epsilon_{R,t} \\ \epsilon_{z,t} \end{bmatrix}. \tag{12}$$

Reference: Sims (1980).

Long-run Restrictions

- Now suppose that

$$y_t = \begin{bmatrix} \text{Inflation} \\ \\ \text{Output Growth} \end{bmatrix}, \quad \epsilon_t = \begin{bmatrix} \epsilon_{R,t} \\ \\ \epsilon_{z,t} \end{bmatrix} = \begin{bmatrix} \text{Monetary Policy Shock} \\ \\ \text{Technology Shock} \end{bmatrix}$$

Moreover,

$$y_t = (\sum_{j=0}^{\infty} C_j L^j) u_t = C(L) u_t. \tag{13}$$

Consider the following assumption: monetary policy shocks do not raise output in the long-run.

Long-run Restrictions

- Let's examine the moving average representation of y_t in terms of the structural shocks

$$
\begin{aligned}
y_t &= \begin{bmatrix} c_{11}(L) & c_{12}(L) \\ c_{21}(L) & c_{22}(L) \end{bmatrix} \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \epsilon_{R,t} \\ \epsilon_{z,t} \end{bmatrix} \\
&= \begin{bmatrix} \cdot & \cdot \\ a_{11}\cos\varphi c_{21}(L) + (a_{21}\cos\varphi + a_{22}\sin\varphi)c_{22}(L) & \cdot \end{bmatrix} \begin{bmatrix} \epsilon_{R,t} \\ \epsilon_{z,t} \end{bmatrix} \\
&= \begin{bmatrix} d_{11}(L) & d_{12}(L) \\ d_{21}(L) & d_{22}(L) \end{bmatrix} \begin{bmatrix} \epsilon_{R,t} \\ \epsilon_{z,t} \end{bmatrix}
\end{aligned}
$$

- Suppose that in period $t = 0$ log output and log prices are equal to zero. Then the log-level of output and prices in period $t = T > 0$ is given by

$$
y_T^c = \sum_{t=1}^{T} y_t = \sum_{t=1}^{T} \sum_{j=0}^{\infty} D_j \epsilon_{t-j} \tag{14}
$$

Long-run Restrictions

- Now consider the derivative
$$\frac{\partial y_T^c}{\partial \epsilon_1'} = \sum_{j=0}^{T-1} D_j \tag{15}$$

- Letting $T \longrightarrow \infty$ gives us the long-run response of the level of prices and output to the shock ϵ_1:
$$\frac{\partial y_\infty^c}{\partial \epsilon_1'} = \sum_{j=0}^{\infty} D_j = D(1) \tag{16}$$

- Here, we want to restrict the long-run effect of monetary policy shocks on output:
$$d_{21}(1) = 0 \tag{17}$$

Long-run Restrictions

- This leads us to the equation

$$[a_{11}c_{21}(1) + a_{21}c_{22}(1)] \cos\varphi + a_{22}c_{22}(1) \sin\varphi = 0. \tag{18}$$

- Notice that the equation has two solutions for $\varphi \in (-\pi, \pi]$. Under one solution a positive monetary policy shock is contractionary, under the other solution it is expansionary. The shape of the responses is, of course, the same.

- Reference: Blanchard and Quah (1989).

Sign Restrictions

- Again consider

$$y_t = \begin{bmatrix} \text{Inflation} \\ \text{Output Growth} \end{bmatrix}, \quad \epsilon_t = \begin{bmatrix} \epsilon_{R,t} \\ \epsilon_{z,t} \end{bmatrix} = \begin{bmatrix} \text{Monetary Policy Shock} \\ \text{Technology Shock} \end{bmatrix}$$

- Our identification assumption is: upon impact, a monetary policy shock raises both prices and output. It can be verified that

$$\frac{\partial y_t}{\partial \epsilon_{R,t}} = \begin{bmatrix} a_{11}\cos\varphi + (a_{21}\cos\varphi + a_{22}\sin\varphi) \\ a_{11}\cos\varphi + (a_{21}\cos\varphi + a_{22}\sin\varphi) \end{bmatrix}. \tag{19}$$

Sign Restrictions

- Thus, we obtain the sign restrictions

$$
\begin{aligned}
0 &< a_{11} \cos \varphi + (a_{21} \cos \varphi + a_{22} \sin \varphi) \\
0 &< a_{11} \cos \varphi + (a_{21} \cos \varphi + a_{22} \sin \varphi)
\end{aligned}
$$

  which restrict φ to be in a certain subset of $(-\pi, \pi]$ and will generate a range of responses.

- References: Canova and De Nicolo (2002), Faust (1998), Uhlig (2005).

Impulse Responses and Variance Decompositions

- Impulse responses are defined as

$$\frac{\partial y_{t+h}}{\partial \epsilon_t'} = C_h \Phi_\epsilon \tag{20}$$

and correspond to the MA coefficient matrices in the moving average representation of y_t in terms of structural shocks.

Impulse Responses and Variance Decompositions

- The covariance matrix of y_t is given by

$$\Gamma_{yy,0} = \sum_{j=0}^{\infty} C_j \Phi_\epsilon \mathcal{I} \Phi_\epsilon' C_j' \tag{21}$$

Let $\mathcal{I}^i$ be matrix for which element i, i is equal to one and all other elements are equal to zero. Then we can define the contribution of the i'th structural shock to the variance of y_t as

$$\Gamma_{yy,0}^{(i)} = \sum_{j=0}^{\infty} C_j \Phi_\epsilon \mathcal{I}^{(i)} \Phi_\epsilon' C_j' \tag{22}$$

Thus the fraction of the variance of $y_{l,t}$ explained by shock i is

$$\frac{[\Gamma_{yy,0}^{(i)}]_{ll}}{[\Gamma_{yy,0}]_{ll}}. \tag{23}$$

Alternative Setups

- Sims and Zha (1998) and subsequent work start out from the specification

$$A_0 y_t = A_1 y_{t-1} + \dots A_p y_{t-p} + \epsilon_t$$

  where ϵ_t's are structural shocks.

- Analyze the model directly in terms of A matrices.

- Impose identification restrictions on A_0.

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