Autocovariances and Impulse Response Functions of a DSGE Model

Frank Schorfheide

Department of Economics, University of Pennsylvania

State-Space Representation

- Log-linearized DSGE models can be written as state-space models:

$$\begin{aligned} \text{measurement} &: \; y_t = A(\theta) + B(\theta)s_t & (1) \\ \text{state transition} &: \; s_t = \Phi_1(\theta)s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t. & (2) \end{aligned}$$

- Make distributional assumption: $\epsilon_t \sim iid\mathcal{N}(0, \Sigma_\epsilon(\theta))$.

- It is only assumed that the y_t's are observable. The vector s_t may have unobservable elements such as conditional expectations or a latent productivity process.

- We obtained the state transition equation when we solved the LRE model.

- Note that s_t evolves according to VAR(1) and y_t is a linear function of s_t

Autocovariances

- We begin by calculating the autocovariance function of s_t.
- We assume that s_t is covariance stationary, which requires that all eigenvalues of the matrix Φ_1 are less than one in absolute value.
- If the eigenvalues of Φ_1 are all less than one in absolute values and the VAR was initialized in the infinite past, then the expected value is given by $I\!\!E[s_t] = 0$.
- Moreover, the autocovariance matrix of order zero has to satisfy the equation

$$\Gamma_{ss,0} = I\!\!E[s_t s_t'] = \Phi_1 \Gamma_{ss,0} \Phi_1' + \Phi_\epsilon I\!\!E[\epsilon_t \epsilon_t'] \Phi_\epsilon' \tag{3}$$

Autocovariances

- **Definition:** Let A and B be 2×2 matrices with the elements

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

The *vec* operator is defined as the operator that stacks the columns of a matrix, that is,

$$vec(A) = [a_{11}, a_{21}, a_{12}, a_{22}]'$$

and the Kronecker product is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} \quad \Box$$

- **Lemma** Let A, B, C be matrices whose dimension are such that the product ABC exists. Then

$$vec(ABC) = (C' \otimes A)vec(B) \quad \Box$$

Autocovariances

- A closed form solution for the elements of the covariance matrix of s_t can be obtained as follows

$$\begin{aligned} vec(\Gamma_{ss,0}) &= (\Phi_1 \otimes \Phi_1) vec(\Gamma_{ss,0}) + vec(\Phi_\epsilon I\!\!E[\epsilon_t \epsilon_t'] \Phi_\epsilon') \\ &= [I - (\Phi_1 \otimes \Phi_1)]^{-1} vec(\Phi_\epsilon I\!\!E[\epsilon_t \epsilon_t'] \Phi_\epsilon') \end{aligned} \tag{4}$$

  Since

$$I\!\!E[s_t s_{t-h}'] = \Phi_1 I\!\!E[s_{t-1} s_{t-h}'] + \Phi_\epsilon I\!\!E[\epsilon_t \epsilon_{t-h}'] \Phi_\epsilon' \tag{5}$$

  we can deduce that

$$\Gamma_{ss,h} = \Phi_1^h \Gamma_{ss,0} \tag{6}$$

- Notice that $\Gamma_{ss,-h} = \Gamma_{ss,-h}'$.

Autocovariances

- Finally, using $y_t = A(\theta) + B(\theta)s_t$, we deduce
- $I\!E[y_t] = A(\theta)$ and
- $\Gamma_{yy,h} = B(\theta)\Gamma_{ss,h}B(\theta)'$.

Impulse Response Functions

- We can express s_t as a $MA(\infty)$ of ϵ_t:

$$s_t = \sum_{j=0}^{\infty} \Phi_1^j(\theta)\epsilon_t.$$

- Hence,

$$\frac{\partial s_t}{\partial \epsilon'_{t-j}} = \Phi_1^j(\theta)$$

- Moreover, using the measurement equation

$$\frac{\partial y_t}{\partial \epsilon'_{t-j}} = B(\theta)\Phi_1^j(\theta)$$

  which provides us the response of y_t to a shock that occurred j periods ago.